Continuously Increasing Subsequences of Random Multiset Permutations

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Abstract

For a word π and an integer i, we define $L(\pi)$ to be the length of the longest subsequence of the form $i(i+1)\cdots j$ for some i, and we let $L^1(\pi)$ be the length of the longest such subsequence beginning with 1. In this paper, we estimate the expected values of $L^1(\pi)$ and $L(\pi)$ when π is chosen uniformly at random from all words which use each of the first n positive integers exactly m times. We show that $\mathbb{E}[L^1(\pi)] \sim m$ if n is sufficiently large in terms of m as m tends towards infinity, confirming a conjecture of Diaconis, Graham, He, and Spiro. We also show that $\mathbb{E}[L(\pi)]$ is asymptotic to the inverse gamma function $\Gamma^{-1}(n)$ if n is sufficiently large in terms of m as m tends towards infinity.

1 Introduction

1.1 Main Results

Given integers m and n, let $\mathfrak{S}_{m,n}$ denote the set of words π which use each integer in $[n] := \{1, 2, \ldots, n\}$ exactly m times. We will refer to $\pi \in \mathfrak{S}_{m,n}$ as a *multiset permutation*. For example, $\pi = 211323 \in \mathfrak{S}_{2,3}$. We begin by studying the expected length of the longest increasing subsequence of the form $123\cdots j$ in a random permutation $\pi \in \mathfrak{S}_{m,n}$. We focus on the regime where n is much larger than m, as in the opposite regime, this length will be equal to its maximum size n with high probability.

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This question was motivated by work of Diaconis, Graham, He, and Spiro [DGHS21] who studied this quantity due to its relationship with a certain card game that we describe later in this paper. They proved that this expected length is at most $m+Cm^{3/4}\log m$ for some absolute constant C provided n is sufficiently large in terms of m. It was conjectured in [DGHS21] that this upper bound of m is asymptotically tight for n sufficiently large in terms of m. Heuristically, such a bound comes from the fact that you expect the first 1 of π to lie near position n, then the first 2 after this 1 to appear about n positions after that, and so on, giving a sequence of length about m.

It turns out that this conjecture of [DGHS21] holds, and much beyond initial expectations, we are able to find an exact formula for the limiting expectation for any fixed m, as well as a precise estimate of this value as m tends towards infinity. To state this result, we let $L^1_{m,n}(\pi)$ denote the length of the longest subsequence of the form $12\cdots j$ in π. For example, if $\pi = 2341524315$ then $L^1_{2,5}(\pi) = 3$.

Theorem 1.1.

(a) *For any integer $m \geq 1$, let $\alpha_1, \ldots, \alpha_m$ be the zeroes of $E_m(x) := \sum_{k=0}^{m} \frac{x^k}{k!}$. If $\pi \in \mathfrak{S}_{m,n}$ is chosen uniformly at random, then*

$$\mathcal{L}^1_m := \lim_{n\to\infty} \mathbb{E}[L^1_{m,n}(\pi)] = -1 - \sum \alpha_i^{-1} e^{-\alpha_i}. \tag{1}$$

(b) *There exists an absolute constant $\beta > 0$ such that*

$$\mathcal{L}^1_m = m + 1 - \frac{1}{m+2} + O(e^{-\beta m}).$$

For example, when $m = 1$ we have $E_1(x) = 1 + x$ and $\alpha_1 = -1$, implying $\mathcal{L}^1_1 = -1 + e$, which can also be proven by elementary means. For $m = 2$ we have $E_2(x) = 1 + x + x^2/2$ and $\alpha_1 = -1 - i, \alpha_2 = -1 + i$. From this, Theorem 1.1(a) gives the closed form expression

$$\mathcal{L}^1_2 = e\left(\cos(1) + \sin(1)\right) - 1,$$

which does not seem to be provable by elementary means.

We note that even to prove a much weaker estimate such as $\mathcal{L}^1_m = m + O(1)$, the only proof we know of requires deducing the full strength of Theorem 1.1, i.e. computing the *exact* formula for $\mathcal{L}^1_m$ in part (a), and then deriving from it the accurate asymptotics for $\mathcal{L}^1_m$ in part (b).

See Section 4.1 for further discussions on the limiting distribution of $L^1_{m,n}(\pi)$ as $n \to \infty$, beyond just its expectation.

Our next result gives precise bounds for the length of a longest continuously increasing subsequence in a random permutation of $\mathfrak{S}_{m,n}$ which does not necessarily start at 1. To this end, we let $L_{m,n}(\pi)$ denote the length of a longest increasing subsequence of π of the form $i(i+1)(i+2)\cdots j$ for some i. For example, if $\pi = 2341524315$ then $L_{2,5}(\pi) = 4$ due to the subsequence 2345.

To state our result in this setting, we recall that the gamma function $\Gamma(x)$ is a function which, in particular, gives a bijection from $x \geq 1$ to $y \geq 1$ and which satisfies $\Gamma(n) = (n-1)!$ for non-negative integers n. We denote the inverse of this bijection by $\Gamma^{-1}(y)$, which is known to be asymptotic to $\log(y)/\log\log(y)$ as y tends to infinity.

Theorem 1.2. *There exists an absolute constant $C > 0$ and a sufficiently fast growing function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\left|\mathbb{E}[L_{m,n}(\pi)] - \Gamma^{-1}(n)\right| \leq C\left(1 + \frac{\log m}{\log(\Gamma^{-1}(n))}\Gamma^{-1}(n)\right)$$

for any $m \geq 1$ and $n \geq f(m)$.

In particular, for any $m \geq 1$, we have

$$\lim_{n\to\infty} \frac{\mathbb{E}[L_{m,n}(\pi)]}{\Gamma^{-1}(n)} = 1.$$

A careful reading of our proof shows that one can take $f(m) = \exp(\exp(K \log^2 m))$ for some sufficiently large constant K. We note that when $m = 1$, the error term of Theorem 1.2 is $\Theta(1)$, but for $m \geq 2$ it is $\Theta(\frac{\log m}{\log \Gamma^{-1}(n)}\Gamma^{-1}(n))$, which is fairly close to the main term of $\Gamma^{-1}(n)$. Thus the behavior of $\mathbb{E}[L_{m,n}(\pi)]$ changes somewhat dramatically as soon as one starts to consider multiset permutations as opposed to just permutations.

1.2 History and Related Work

Determining $L^i_{m,n}(\pi)$ and $L_{m,n}(\pi)$ can be viewed as variants of the well-studied problem of determining the length of the longest increasing subsequence in a random permutation of length n, and we denote this quantity by $\widetilde{L}_n$. It was shown by Logan and Shepp [LS77] and Vershick and Kerov [VK77] that $\mathbb{E}[\widetilde{L}_n] \sim 2\sqrt{n}$, answering a famous problem of Ulam. Later Baik, Deift, and Johansson [BDJ99] showed that the limiting distribution of $\widetilde{L}_n$ is the Tracy-Widom distribution. Some work with the analogous problem for multiset permutations has been considered recently by Al-Meanazel and Johnson [AMJ20]. Much more can be said about this topic, and we refer the reader to the excellent book by Romik [Rom15] for more information.

The initial motivation for studying $L^1(\pi)$ was due to its relationship to a card guessing experiment introduced by Diaconis and Graham [DG81]. To start the experiment, one shuffles a deck of mn cards which consists of n distinct card types each appearing with multiplicity m. In each round, a subject iteratively guesses what the top card of the deck is according to some strategy G. After each guess, the subject is told whether their guess was correct or not, the top card is discarded, and then the experiment continues with the next card. This experiment is known as the *partial feedback model*. For more on the history of the partial feedback model we refer the reader to [DGS21].

If G is a strategy for the subject in the partial feedback model and $\pi \in \mathfrak{S}_{m,n}$, we let $P(G, \pi)$ denote the number of correct guesses made by the subject if they follow strategy G and the deck is shuffled according to π. We say that G is an *optimal strategy* if $\mathbb{E}[P(G, \pi)] = \max_{G'} \mathbb{E}[P(G', \pi)]$, where G' ranges over all strategies and $\pi \in \mathfrak{S}_{m,n}$ is chosen uniformly at random. Optimal strategies are unknown in general, and even if they were known they would likely be too complex for a human subject to implement in practice. As such there is interest in coming up with (simple) strategies G such that $\mathbb{E}[P(G, \pi)]$ is relatively large.

One strategy is the *trivial strategy* which guesses card type 1 every single round, guaranteeing a score of exactly m at the end of the experiment. A slightly better strategy is the *safe strategy* G_{safe} which guesses card type 1 every round until all m are guessed correctly, then 2's until all m are guessed correctly, and so on. It can be deduced from arguments given by Diaconis, Graham, and Spiro [DGS21] that $\mathbb{E}[P(G_{safe}, \pi)]$ is $m+1-\frac{1}{m+1}$ plus an exponential error term, so the safe strategy does just a little better than the trivial strategy.

Another natural strategy is the *shifting strategy* G_{shift}, defined by guessing 1 until you are correct, then 2 until you are correct, and so on; with the strategy being defined arbitrarily in the (very rare) event that one correctly guesses a copy of each card type. It is not difficult to see that $P(G_{shift}, \pi) \geq L^1_{m,n}(\pi)$, with equality holding provided the player does not correctly guess n. Thus Theorem 1.1(b) shows that the expected number of correct guesses under the shifting strategy is close to $m+1-\frac{1}{m+2}$, which is slightly better than the trivial strategy, and very slightly better than the safe strategy.

1.3 Preliminaries

We let $[n] := \{1, 2, \ldots, n\}$ and let $[m]^n$ be the set of tuples of length n with entries in $[m]$. Whenever we write, for example, $\Pr[L_{m,n}(\pi) \geq k]$, we will assume π is chosen uniformly at random from $\mathfrak{S}_{m,n}$ unless stated otherwise.

Throughout this paper we use several basic results from probability theory. One such result is that if X is a non-negative integer-valued random variable, then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr[X \geq k].$$

A crucial observation that we use throughout the text is the following.

Observation 1.3. *For $k \leq n$, if $\pi \in \mathfrak{S}_{m,n}$ and $\tau \in \mathfrak{S}_{m,k}$ are drawn uniformly at random, then*

$$\Pr[L^1_{m,n}(\pi) \geq k] = \Pr[L^1_{m,k}(\tau) = k].$$

Proof. For $\pi \in \mathfrak{S}_{m,n}$, let $\Phi_x(\pi) \in \mathfrak{S}_{m,k}$ be the word obtained by deleting every letter from π which is larger than k. Note that $L^1_{m,n}(\pi) \geq k$ if and only if $L^1_{m,k}(\Phi_x(\pi)) = k$. Moreover, it is not difficult to see that $\Phi_x(\pi)$ is distributed uniformly at random in $\mathfrak{S}_{m,k}$ provided π is distributed uniformly at random in $\mathfrak{S}_{m,n}$, proving the result. □

2 Proof of Theorem 1.1

2.1 Theorem 1.1(a): Generating Functions

We say that a word $\pi \in \mathfrak{S}_{m,n}$ has a *complete increasing subsequence* if $L^1_{m,n}(\pi) = n$. Let $h_m(n)$ be the number of words $\pi \in \mathfrak{S}_{m,n}$ which have a complete increasing subsequence. Horton and Kurn [HK81, Corollary (c)] give the following formula for $h_m(n)$.

Theorem 2.1 ([HK81]). *The number of words $\pi \in \mathfrak{S}_{m,n}$ which have a complete increasing subsequence, $h_m(n)$, is given by*

$$h_m(n) = \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \frac{(mn)!}{l!} \frac{(-1)^{l-n}}{\prod_{j=1}^m (m-j)!^{i_j}},$$

where

$$l = \sum_{j=1}^m j i_j,$$

$\mathcal{N}(n,m)$ is the set of weak compositions of n into m parts, i.e.,

$$\mathcal{N}(n,m) \coloneqq \left\{ (i_1,\ldots,i_m) \in \mathbb{Z}_{\geq 0}^m \middle| \sum_{j=1}^m i_j = n \right\},$$

and

$$\binom{n}{i_1,\ldots,i_m} = \frac{n!}{\prod_{j=1}^m i_j!}$$

is a multinomial coefficient.

Notice that $\mathcal{L}_m^1$ can be expressed in terms of $h_m(n)$ as follows:

$$\mathcal{L}_m^1 = \lim_{k\to\infty} \mathbb{E}[L_{m,k}^1(\pi)] = \lim_{k\to\infty} \sum_{n=1}^k \Pr[L_{m,k}^1(\pi) \geq n] = \lim_{k\to\infty} \sum_{n=1}^k \frac{h_m(n)}{|\mathfrak{S}_{m,n}|} = \sum_{n=1}^\infty \frac{h_m(n)}{|\mathfrak{S}_{m,n}|}, \tag{2}$$

where the third equality is due to Observation 1.3. Note that $|\mathfrak{S}_{m,n}| = (mn)!/(m!)^n$. Thus, as a consequence of Theorem 2.1, we have

$$\frac{h_m(n)}{|\mathfrak{S}_{m,n}|} = (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \frac{1}{\prod_{j=1}^m (m-j)!^{i_j}} \frac{(-1)^l}{l!}, \tag{3a}$$

$$= (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \prod_{j=1}^m \left(\frac{(-1)^j}{(m-j)!}\right)^{i_j} \frac{1}{l!}. \tag{3b}$$

Intuitively, if the $1/l!$ were removed from the right-hand-side expression in (3b), then by using the multinomial theorem we could write this expression as an n^{th} power, turning (2) into a geometric series. The next few paragraphs formalize this idea.

We begin by replacing $(-1)^l$ by x^l in the right-hand-side of (3a) to obtain the polynomial

$$\begin{aligned} p_{m,n}(x) \coloneqq& (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \frac{1}{\prod_{j=1}^m (m-j)!^{i_j}} \frac{x^l}{l!} \\ =& (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \prod_{j=1}^m \left(\frac{x^j}{(m-j)!}\right)^{i_j} \frac{1}{l!}. \end{aligned} \tag{4}$$

Thus,

$$p_{m,n}(-1) = \frac{h_m(n)}{|\mathfrak{S}_{m,n}|}.$$

Next, we define an operator in order to remove the $l!$ from the denominator. Let R be a commutative ring containing $\mathbb{Q}$ and x be an indeterminate. Let $\Phi_x : R[[x]] \to R[[x]]$ be an R-linear operator[1] defined termwise on monomials by

$$\Phi_x(x^n) = \frac{x^n}{n!}.$$

Throughout this article, R is either $\mathbb{C}$ or $\mathbb{C}[[y]]$ for an indeterminate y, and we shall refer to this R-linear map as Φ_x in both cases. Notice that Φ_x is invertible for any such ring R. A key property that we use about Φ_x is

$$\Phi_x\left(\frac{1}{1-ax}\right) = \Phi_x\left(\sum_{i=0}^{\infty}(ax)^i\right) = \sum_{i=0}^{\infty}\frac{(ax)^i}{i!} = e^{ax}. \tag{5}$$

Consider the polynomial

$$\begin{aligned} q_{m,n}(x) \coloneqq {\Phi_x}^{-1}\left(p_{m,n}(x)\right) &= (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \frac{x^l}{\prod_{j=1}^m (m-j)!^{i_j}} \\ &= (-m!)^n \sum_{(i_1,\ldots,i_m)\in\mathcal{N}(n,m)} \binom{n}{i_1,\ldots,i_m} \prod_{j=1}^m \left(\frac{x^j}{(m-j)!}\right)^{i_j}. \end{aligned}$$

Notice that,

$$q_{m,n}(x) = \left(-m!\sum_{j=1}^m \frac{x^j}{(m-j)!}\right)^n = (q_{m,1}(x))^n.$$

Let $P_m(x,y)$ and $Q_m(x,y)$ be the ordinary generating functions of $p_{m,n}(x)$ and $q_{m,n}(x)$ respectively, i.e.

$$P_m(x,y) \coloneqq \sum_{n=0}^{\infty} p_{m,n}(x)y^n,$$

$$Q_m(x,y) \coloneqq \sum_{n=0}^{\infty} q_{m,n}(x)y^n = {\Phi_x}^{-1}\left(P_m(x,y)\right).$$

Putting everything together, we see that

$$\mathcal{L}_m^1 = P_m(-1,1) - 1. \tag{6}$$

and thus it suffices to find a nice closed form expression for $P_m(x,y)$. Note that

$$q_{m,1}(x) = -m!x^m E_{m-1}(1/x),$$

[1] This operator is often called the formal Laplace transform with respect to x.

where we recall the polynomial $E_{m-1}(x)$ is defined in Theorem 1.1 by $E_{m-1}(x) = \sum_{k=0}^{m-1} x^k/k!$. As $q_{m,n}(x) = (q_{m,1}(x))^n$, we have

$$Q_m(x,y) = \frac{1}{1 - yq_{m,1}(x)} = \frac{1}{1 + m!x^m y E_{m-1}(1/x)}. \tag{7}$$

Hence,

$$P_m(x,y) = \Phi_x\left(Q_m(x,y)\right) = \Phi_x\left(\frac{1}{1 + m!x^m y E_{m-1}(1/x)}\right), \tag{8}$$

and thus

$$P_m(x,1) = \Phi_x\left(\frac{1}{1 + m!x^m E_{m-1}(1/x)}\right) = \Phi_x\left(\frac{1}{m!x^m E_m(1/x)}\right). \tag{9}$$

We now prove the main result of this subsection.

Proposition 2.2. *Let $\alpha_1, \ldots, \alpha_m$ be the zeroes of the polynomial $E_m(x)$. The formal power series $P_m(x,1)$ satisfies*

$$P_m(x,1) = -\sum_{i=1}^{m} \alpha_i^{-1} e^{\alpha_i x}.$$

Proof. Let $g(x) \coloneqq m!x^m E_m(1/x)$. Since $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$ are the zeroes of $g(x)$, we have

$$g(x) = m!(x - \alpha_1^{-1}) \cdots (x - \alpha_m^{-1}).$$

Notice that $E_m(x)$ has no repeated zeroes. This is true because, if α is a repeated zero of $E_m(x)$, it is also a zero of its derivative $E_m'(x) = E_{m-1}(x)$. But then α has to be a zero of $E_m(x) - E_{m-1}(x) = x^m/m!$, which is only possible if $\alpha = 0$, a contradiction as 0 is not a zero of $E_m(x)$.

Thus $\alpha_1, \ldots, \alpha_m$ are pairwise distinct, and hence the zeroes of $g(x)$, being $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$, are also pairwise distinct. This, together with (7), implies that $Q_m(x,1)$ has the partial fraction decomposition

$$Q_m(x,1) = \frac{1}{g(x)} = \sum_{i=1}^{m} \frac{1}{g'\left(\alpha_i^{-1}\right)} \cdot \frac{1}{x - \alpha_i^{-1}}.$$

The derivative of g is

$$g'(x) = m!\left(\frac{mx^m E_m(1/x)}{x} - \frac{x^m E_m'(1/x)}{x^2}\right) = m!\left(\frac{mx^m E_m(1/x)}{x} - \frac{x^m(E_m(1/x) - x^{-m}/m!)}{x^2}\right)$$

Hence for any i,

$$g'\left(\alpha_i^{-1}\right) = \alpha_i^2$$

which gives

$$P_m(x,1) = \Phi_x\left(Q_m(x,1)\right) = -\sum_{i=1}^{m} \Phi_x\left(\frac{1}{\alpha_i(1-\alpha_i x)}\right) = -\sum_{i=1}^{m} \alpha_i^{-1} e^{\alpha_i x},$$

where this last step used (5). ∎

This proposition, together with (6), implies Theorem 1.1(a).

2.2 Theorem 1.1(b): Exponential Sums of Zeroes

We remind the reader that Theorem 1.1(b) claims

$$\left|\mathcal{L}_m^1 - \left(m+1-\frac{1}{m+2}\right)\right| \le O(e^{-\beta m})$$

for some constant $\beta > 0$. It turns out that Theorem 1.1(a) allows for an elementary proof of this claim. We do this by showing that $\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i} = -m-2+\frac{1}{m+2}+O(e^{-\beta m})$ for some positive constant β. We follow the approach used by Conrey and Ghosh [CG88], where a similar exponential sum was estimated. To avoid unnecessary complications, we first give an informal proof that captures all of the essential algebra needed to compute the asymptotics of $\sum_i \alpha_i^{-1} e^{-\alpha_i}$, and then we make this analysis rigorous by using an idea of Conrey and Ghosh. We point out that our method actually works for higher degree exponential sums. That is, we can prove that for any integer $d \ge 0$, there is an "effectively computable" rational function $F_d(m) \in \mathbb{Q}(m)$ such that

$$\sum_{i=1}^{m} \alpha_i^{-d} e^{-\alpha_i} = F_d(m) + O(e^{-\beta_d m}) \tag{10}$$

for some constant $\beta_d > 0$. Section 2.2.1 contains all the necessary ingredients to construct an algorithm to compute $F_d(m)$, though it only addresses the relevant special case $d = 1$. The work of Conrey and Ghosh [CG88] gives the case $d = 0$. The precise expression of $F_d(m)$ for $d \ge 2$ may be useful if one wishes to estimate higher moments of $L_{m,n}^1(\pi)$, but it will not be used in any result of this paper.

Though not relevant to the proof, we recall the large-m asymptotic distribution of the α_i, namely, the zeroes of $E_m(x) = \sum_{k=0}^{m} x^k/k!$. By a result of Szegő (see Zemyan [Zem05, p. 895] for references and an illustration), as m goes to ∞, the scaled zeroes α_i/m (noting that α_i depends on m) are asymptotic to the portion of the curve $\{z \in \mathbb{C} : |ze^{1-z}| = 1\}$ inside the unit disc (this is often called the Szegő curve). The scaled zeroes for $m = 60$ have been plotted in Figure 1 along with the Szegő curve. Also see [?] for an interactive demonstration.

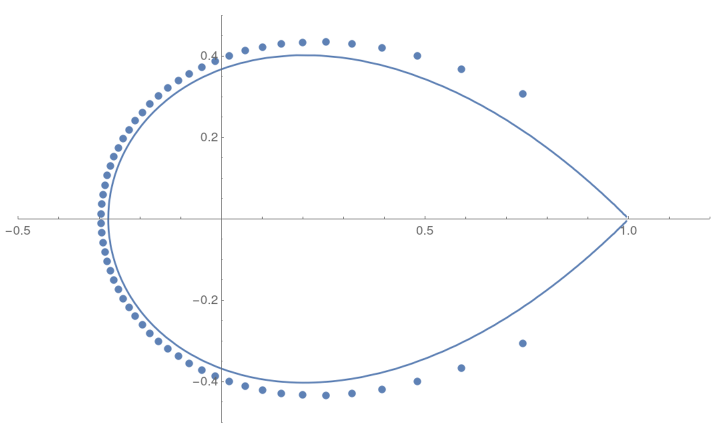

Figure 1: The zeroes of $E_m(x) = \sum_{k=0}^{m} x^k/k!$, divided by m, have been plotted for $m = 60$. See how they approach the curve $\{z \in \mathbb{C} : |ze^{1-z}| = 1\}$ inside the unit disc (Szegő curve).

In particular, $\sum \alpha_i^{-1} e^{-\alpha_i}$ contains terms that are exponentially large in m (such as the α_i's with the most negative real parts), so the desired asymptotic $\sum_i \alpha_i^{-1} e^{-\alpha_i} = -m - 2 + \frac{1}{m+2} + O(e^{-\beta m})$ is unexpected in some sense. As such, algebraic methods are necessary to take into account the fact that the α_i are the set of *all* zeroes of an explicit polynomial $E_m(x)$.

The method of Conrey and Ghosh can be summarized into the following two key points. First, the sum $\sum \alpha_i^{-t}$ for any integer $t \geq 1$ is a symmetric rational function, so it can in principle be expressed in terms of coefficients of $E_m(x)$ (see Proposition 2.3(c) below). The vanishing of $\sum \alpha_i^{-t}$ for a wide range of t (namely, $2 \leq t \leq m$) will be the key reason behind the unusually small asymptotics. Second, the fact that $E_m(\alpha_i) = 0$ allows us to rewrite $e^{-\alpha_i}$ as

$$\frac{1}{e^{\alpha_i}} = \frac{1}{e^{\alpha_i} - E_m(\alpha_i)} = \frac{1}{R_m(\alpha_i)},$$

where

$$R_m(x) := e^x - E_m(x) = \sum_{k=m+1}^{\infty} \frac{x^k}{k!}.$$

Replacing $1/R_m(x)$ by its Laurent expansion at $x = 0$, we are able to rewrite $\alpha_i^{-1} e^{-\alpha_i}$ as an (infinite) linear combination of α_i^{-t}, where $t \in \mathbb{Z}$ and $t \leq m + 2$. It turns out that the terms with $t \leq 1$ will only contribute to an exponential decay to the asymptotics (essentially from the estimate in Lemma 2.5 below), so we get the main terms $-m - 2 + \frac{1}{m+2}$ after working out the contribution of $\sum \alpha_i^{-t}$ for $1 \leq t \leq m + 2$.

2.2.1 Informal Computation

We now "compute" the large-m asymptotics of $\sum \alpha_i^{-1} e^{-\alpha_i}$, pretending that the following two assumptions hold:

- The Laurent expansion of $1/R_m(x)$ at $x = 0$ converges at $x = \alpha_i$ for all α_i.
- If $t \leq 1$, the terms involving α_i^{-t} can be neglected due to their small coefficient.

See Section 2.2.2 for a rigorous treatment to work around the false assumptions above.

We use an exact formula of Zemyan [Zem05] (see Proposition 2.3 below): if $m \geq 1$ and α_i $(1 \leq i \leq m)$ are the roots of $E_m(x)$, then

$$\sum_{i=1}^{m} \alpha_i^{-t} = \begin{cases} -1 & t = 1, \\ 0 & 2 \leq t \leq m, \\ \dfrac{(-1)^k}{k!m!} & t = m+1+k, 0 \leq k \leq m. \end{cases} \tag{11}$$

Recall $R_m(x) = \sum_{k=m+1}^{\infty} \frac{x^k}{k!}$. We expand the first few terms of $1/R_m(x)$ by hand:

$$\begin{aligned}\frac{1}{R_m(x)} &= \frac{1}{\sum_{k=m+1}^{\infty} x^k/k!} = \frac{(m+1)!}{x^{m+1}} \frac{1}{1 + \sum_{k=m+2}^{\infty} x^{k-m-1}(m+1)!/k!} \\ &= \frac{(m+1)!}{x^{m+1}} \frac{1}{1 + x/(m+2) + O(x^2)} \\ &= \frac{(m+1)!}{x^{m+1}} \left(1 - \frac{x}{m+2} + O(x^2)\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i} &= \sum_{i=1}^{m} \frac{1}{\alpha_i R_m(\alpha_i)} \\ &= \sum_{i=1}^{m} \left((m+1)!\, \alpha_i^{-(m+2)} - \frac{(m+1)!}{m+2} \alpha_i^{-(m+1)} + \sum_{t \le m} c_t \alpha_i^{-t} \right) \\ &= (m+1)! \sum_{i=1}^{m} \alpha_i^{-(m+2)} - \frac{(m+1)!}{m+2} \sum_{i=1}^{m} \alpha_i^{-(m+1)} + \sum_{t \le m} c_t \sum_{i=1}^{m} \alpha_i^{-t},\end{aligned}$$

where c_t $(t \le m)$ are the rest of the coefficients of the Laurent expansion of $1/(xR_m(x))$ at $x = 0$.

By (11), $\sum_{i=1}^{m} \alpha_i^{-t} = 0$ for $2 \le t \le m$, and the assumption says that c_t is small enough for $t \le 1$ so that $\sum_{t \le 1} c_t \sum_{i=1}^{m} \alpha_i^{-t}$ is negligible. Hence,

$$\begin{aligned}\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i} &\approx (m+1)! \sum_{i=1}^{m} \alpha_i^{-(m+2)} - \frac{(m+1)!}{m+2} \sum_{i=1}^{m} \alpha_i^{-(m+1)} \\ &= (m+1)! \left(-\frac{1}{m!}\right) - \frac{(m+1)!}{m+2} \frac{1}{m!} \\ &= -m - 1 - \frac{m+1}{m+2} = -m - 2 + \frac{1}{m+2},\end{aligned}$$

finishing the computation of the desired asymptotics of $\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i}$.

2.2.2 Rigorous Analysis

The problem of the previous computation is that the Laurent expansion of $1/R_m(x)$ at $x = 0$ only has a well-controlled tail at $x = \alpha_i$ for a portion of the roots α_i. Conrey and Ghosh resolve this issue by considering α_i with large real part and small real part separately. For the portion of α_i with large real part, we do a direct estimate because $\alpha_i^{-1} e^{-\alpha_i}$ is exponentially small. On the other hand, Figure 1 suggests that α_i with small real part also has absolute value not too large (Proposition 2.3(a) makes this remark precise), so the Laurent expansion of $1/R_m(x)$ is well-controlled at such α_i, and thus we can use the method in Section 2.2.1.

For $1 \le i \le m$, let s_i and t_i be the real and imaginary parts, respectively, of α_i. Let $\gamma^-, \gamma, \gamma^+$ be arbitrary positive numbers such that $0 < \gamma^- < \gamma < \gamma^+ < 1 - \log 2$. We partition $[m]$ into disjoint sets S and L where $i \in S$ when $s_i \le \gamma m$ and $i \in L$ when $s_i > \gamma m$, allowing us to rewrite our desired sum as

$$\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i} = \sum_{i\in S} \alpha_i^{-1} e^{-\alpha_i} + \sum_{i \in L} \alpha_i^{-1} e^{-\alpha_i}.$$

Recall

$$R_m(x) = e^x - E_m(x) = \sum_{k=m+1}^{\infty} \frac{x^k}{k!}.$$

The following results are proven by Conrey and Ghosh [CG88] and Zemyan [Zem05]:

Proposition 2.3.

(a) [CG88, Equations (6) and (7)] *For sufficiently large m, we have $|\alpha_i| \ge me^{\gamma^- -1}$ for $i \in L$ and $|\alpha_i| \le me^{\gamma^+ -1}$ for $i \in S$. Consequently, we have $|\alpha_i| < m/2$ for $i \in S$.*

(b) [CG88, Lemma 1] *For $|x| < \frac{1}{2}(m+2)$, we have*

$$\frac{1}{R_m(x)} = \frac{(m+1)!}{x^{m+1}} \left(1 + \sum_{k=1}^{\infty} c_k x^k \right)$$

with $|c_k| \le \frac{1}{2}(\frac{2}{m+2})^k$.

(c) [Zem05, Theorem 7]

$$\sum_{i=1}^{m} \alpha_i^{-t} = \begin{cases} -1 & t = 1, \\ 0 & 2 \le t \le m, \\ 1/m! & t = m+1, \\ -1/m! & t = m+2. \end{cases}$$

We begin our argument by restricting our attention to the indices in L:

Lemma 2.4.

$$\left| \sum_{i \in L} \alpha_i^{-1} e^{-\alpha_i} \right| \le \gamma^{-1} e^{-\gamma m}.$$

Proof. By the triangle inequality,

$$\left| \sum_{i \in L} \alpha_i^{-1} e^{-\alpha_i} \right| \le \sum_{i \in L} |\alpha_i^{-1} e^{-\alpha_i}|.$$

Note that $|\alpha_i^{-1} e^{-\alpha_i}| = |\alpha_i^{-1}||e^{-s_i}| < |\alpha_i^{-1}| e^{-\gamma m}$. Since $s_i > \gamma m$, we know that $|\alpha_i| > \gamma m$, so $|\alpha_i^{-1}| < (\gamma m)^{-1}$. Thus for $i \in L$, we have that

$$|\alpha_i^{-1} e^{-\alpha_i}| < (\gamma m)^{-1} e^{-\gamma m}.$$

Adding over the elements of L, of which there are at most m, we have $|\sum_{i\in L} \alpha_i^{-1} e^{-\alpha_i}| \leq \gamma^{-1} e^{-\gamma m}$. $\square$

To evaluate the sum for the indices in S, we utilize $R_m(x)$. Since the α_i's are the roots of $E_m(x)$, we have $e^{\alpha_i} = R_m(\alpha_i)$ for $i = 1, \cdots, m$, so $e^{-\alpha_i} = \frac{1}{R_m(\alpha_i)}$.

Lemma 2.5. *For* $|x| < \frac{1}{2}(m+2)$*, we have*

$$\frac{1}{R_m(x)} = \frac{(m+1)!}{x^{m+1}} \left(1 - \frac{x}{m+2} + \sum_{k=2}^{\infty} c_k x^k\right)$$

with $|c_k| \leq \frac{1}{2}(\frac{2}{m+2})^k$.

Proof. All of this follows from Proposition 2.3(b) except for showing that $c_1 = -1/(m+2)$. To show this, we observe by definition of $R_m(x)$ that

$$\frac{x^{m+1}}{(m+1)!R_m(x)} = \left(1 + (m+1)! \sum_{k=1}^{\infty} \frac{x^k}{(m+1+k)!}\right)^{-1}.$$

Conrey and Ghosh [CG88] note that $\left|(m+1)! \sum_{k=1}^{\infty} \frac{x^k}{(m+1+k)!}\right| < 1$ when $|x| < \frac{1}{2}(m+2)$, so this can be expanded as a convergent geometric series in such cases. Thus,

$$\frac{1}{R_m(x)} = \frac{(m+1)!}{x^{m+1}} \left(1 + \sum_{j=1}^{\infty} \left(-(m+1)! \sum_{k=1}^{\infty} \frac{x^k}{(m+1+k)!}\right)^j\right).$$

The only time an x term can appear in the double infinite sum is when $k, j = 1$, so this term has coefficient $-(m+1)!\frac{1}{(m+1+1)!} = \frac{-1}{m+2}$ as desired. $\square$

By Lemma 2.4, we have $\sum_{i=1}^{m} \alpha_i^{-1} e^{-\alpha_i} = \sum_{i\in S} \alpha_i^{-1} e^{-\alpha_i} + O(e^{-\beta m})$ for some $\beta > 0$, so we are left to consider the sum over S. Note that for $i \in S$ we have $|\alpha_i| < m/2$ from Proposition 2.3(a). Thus we can use Lemma 2.5 to conclude that

$$\sum_{i\in S} \alpha_i^{-1} e^{-\alpha_i} = \sum_{i\in S} \frac{\alpha_i^{-1}}{R_m(\alpha_i)} = (m+1)! \sum_{i\in S} \sum_{k=0} c_k \alpha_i^{k-m-2}, \tag{12}$$

where $c_0 = 1$ and $c_1 = \frac{-1}{m+2}$.

Using the values of $\sum_{i=1}^{m} \alpha_i^{-t}$ for $t = 1, \cdots, m+2$ from Proposition 2.3(c) and that $c_0 = 1$ and $c_1 = \frac{-1}{m+2}$, we see that

$$\sum_{k=0}^{m+1}\sum_{i\in S} c_k\alpha_i^{k-m-2} = \sum_{k=0}^{m+1}\left(\sum_{i=1}^{m} c_k\alpha_i^{k-m-2} - \sum_{i\in L} c_k\alpha_i^{k-m-2}\right) = \frac{-1}{m!} - \frac{1}{m!(m+2)} - c_{m+1} - \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2}.$$

Plugging this into (12) gives

$$\begin{aligned}
\sum_{i\in S}\alpha_i^{-1}e^{-\alpha_i} &= (m+1)!\left(\frac{-1}{m!} - \frac{1}{m!(m+2)} - c_{m+1} - \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} + \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right)\\
&= -(m+1) - \frac{m+1}{m+2} - (m+1)!\left(c_{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} - \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right)\\
&= -m-2+\frac{1}{m+2} - (m+1)!\left(c_{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} - \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right).
\end{aligned}$$

The first three terms above match our claimed expression, so it suffices to show that the leftover terms $-(m+1)!\left(c_{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} - \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right)$ are $O(e^{-\beta m})$.

Using the Triangle Inequality, and recalling that $|c_k| \le (\frac{2}{m})^k$ for all k by Proposition 2.3(b), we obtain

$$\begin{aligned}
&\left|c_{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} - \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right|\\
\le&(2/m)^{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L}(2/m)^k|\alpha_i|^{k-m-2} + \sum_{i\in S}\sum_{k=m+2}^{\infty}(2/m)^k|\alpha_i|^{k-m-2}.
\end{aligned}$$

Since $|L|,|S| \le m$, this is at most $(2/m)^{m+1} + m\sum_{k=0}^{m+1}(2/m)^k|\alpha_i|^{k-m-2} + m\sum_{k=m+2}^{\infty}(2/m)^k|\alpha_i|^{k-m-2}$.

Now we make use of Proposition 2.3(a). In the first summation, the quantity $k-m-2$ is negative, so $|\alpha_i|^{k-m-2} \le (me^{\gamma^- -1})^{k-m-2}$. In the second summation, $k-m-2$ is nonnegative, so $|\alpha_i|^{k-m-2} \le (me^{\gamma^+ -1})^{k-m-2}$. Putting this altogether, we have

$$\begin{aligned}
&\left|c_{m+1} + \sum_{k=0}^{m+1}\sum_{i\in L} c_k\alpha_i^{k-m-2} - \sum_{i\in S}\sum_{k=m+2}^{\infty} c_k\alpha_i^{k-m-2}\right|\\
&\le (2/m)^{m+1} + m\sum_{k=0}^{m+1}(2/m)^k(me^{\gamma^- -1})^{k-m-2} + m\sum_{k=m+2}^{\infty}(2/m)^k(me^{\gamma^+ -1})^{k-m-2}\\
&= (2/m)^{m+1} + \frac{m^{-m-1}}{(e^{\gamma^- -1})^{m+2}}\sum_{k=0}^{m+1}(2e^{\gamma^- -1})^k + \frac{m^{-m-1}}{(e^{\gamma^+ -1})^{m+2}}\sum_{k=m+2}^{\infty}(2e^{\gamma^+ -1})^k.
\end{aligned}$$

Note that the first summation is finite and it is bounded above by the convergent infinite sum $\sum_{k=0}^{\infty}(2e^{\gamma^- -1})^k$, which is a constant. Since $0 < \gamma^+ < 1-\log 2$, we have that $2e^{\gamma^+ -1} < 1$, so the second summation also converges. Let C be some constant which serves as an upper bound for both of these summations. The total expression is then at most

$$(2/m)^{m+1} + 2C\frac{m^{-m-1}}{(e^{\gamma^- -1})^{m+2}}.$$

We need to show that this expression will still be $O(e^{-\beta m})$ after we multiply it by $(m+1)!$. By the Stirling approximation, $m!$ is asymptotically $\sqrt{2\pi m}(\frac{m}{e})^m$, so for sufficiently large m, we have

$$(m+1)! = (m+1)m! \leq (m+1)(m-1)(m/e)^m \leq m^2(m/e)^m.$$

Now we examine

$$m^2(m/e)^m \left((2/m)^{m+1} + 2C\frac{m^{-m-1}}{(e^{\gamma^- -1})^{m+2}}\right) = \frac{2^{m+1}}{e^m}m + 2C\frac{m}{e^m(e^{\gamma^- -1})^{m+2}}.$$

Let $D := (e^{\gamma^- -1})^{-1}$. Since $\gamma^- < 1 - \log 2$, we have that $e^{\gamma^- -1} < \frac{1}{2}$ and hence that $D > 2$. However, γ^- can be chosen arbitrarily close to $1 - \log 2$ to ensure $D < e$. In this case

$$\begin{aligned} m^2(m/e)^m \left((2/m)^{m+1} + 2C\frac{m^{-m-1}}{(e^{\gamma^- -1})^{m+2}}\right) &= me^{-m}\left(2^{m+1} + 2CD^{m+2}\right) \\ &= O\left(me^{-m}(2^m + CD^2D^m)\right) \\ &= O\left(me^{-m}D^m\right) \\ &= O\left(e^{m(\frac{\log m}{m} - 1 + \log D)}\right). \end{aligned}$$

Note that $-1 + \log D < 0$ and that the $\frac{\log m}{m}$ is negligible for sufficiently large m, so indeed the sum we are considering is $O(e^{-\beta m})$ for some positive β, completing our proof of Theorem 1.1(b).

3 Proof of Theorem 1.2

At a high level, the proof of Theorem 1.2 revolves around showing that $\Pr[L_{m,n}(\pi) \geq k]$ tends to 0 or 1 depending on if $nm^k/k!$ tends to 0 or infinity as n tends to infinity. The following lemma (which we will apply with t such that $t! \approx n$) will be used to determine the threshold when $nm^k/k!$ shifts from being very small to very large. Here and throughout the text, log denotes the natural logarithm.

Lemma 3.1. *Given integers $m \geq 1$ and $t \geq 2$, let $C > 0$ be a real number such that $k = t + \frac{C \log m}{\log t}t$ is an integer. We have*

$$k! \geq t! \cdot m^{Ct},$$

and if $t \geq m^{10C}$, we have

$$k! \leq t! \cdot (1.1)^k m^{Ct}$$

Here and throughout the text we define the falling factorial $(N)_a := N(N-1)\cdots(N-a+1)$.

Proof. Note that $k! = t!(k)_{k-t}$, so it suffices to show

$$m^{Ct} \le (k)_{k-t} \le (1.1)^k m^{Ct},$$

with the upper bound holding when $t \ge m^{10C}$. When $t \ge m^{10C}$, we have $\log t \ge 10C \log m$, so $k = t + \frac{C \log m}{\log t} t \le 1.1t$. This implies

$$(k)_{k-t} \le k^{k-t} \le (1.1t)^{k-t} \le (1.1)^k \cdot t^{k-t} = (1.1)^k \cdot m^{Ct}.$$

Similarly $(k)_{k-t} \ge t^{k-t} = m^{Ct}$ for all t, proving the result. $\square$

Before delving into the details of the proof, we introduce some auxiliary definitions that will make our arguments somewhat cleaner. The main idea is that we wish to reduce multiset permutations to set permutations by labeling each of the m copies of $i \in [n]$.

To this end, let $\mathfrak{S}^*_{m,n}$ denote the set of permutations of the set $\{i_h : i \in [n],\ h \in [m]\}$. For example, $\tau' := 3_1 2_1 3_2 1_2 2_2 1_1 \in \mathfrak{S}^*_{2,3}$. If $\tau \in \mathfrak{S}^*_{m,n}$ contains a subsequence of the form $(w_1)_{x_1} \cdots (w_k)_{x_k}$, then we will say that τ has a subsequence of type (w, x) where $w = w_1 \cdots w_k$ and $x = x_1 \cdots x_k$. We say that $\tau \in \mathfrak{S}^*_{m,n}$ has a subsequence of type w if it has a subsequence of type (w, x) for some x. For example, τ' defined above has a subsequence of type $(12, 22)$ and hence of type 12, but it contains no subsequence of type 123.

Observation 3.2. *If $\pi \in \mathfrak{S}_{m,n}$ and $\tau \in \mathfrak{S}^*_{m,n}$ are chosen uniformly at random, then for any word w with letters in $[n]$ we have*

$$\Pr[\pi \textit{ contains } w \textit{ as a subsequence}] = \Pr[\tau \textit{ contains a subsequence of type } w].$$

The intuition for this observation is as follows. We can view $\{i_h : i \in [n],\ h \in [m]\}$ as a deck of cards with n card types each having m suits, and we view $\tau \in \mathfrak{S}^*_{m,n}$ as a way of shuffling this deck. The property that τ contains a subsequence of type w is independent of the suits of the cards. Thus if we let $\pi \in \mathfrak{S}_{m,n}$ denote the shuffling τ after ignoring suits, then π contains w as a subsequence if and only if τ contains a subsequence of type w. More formally, one can prove this result by considering the map $\Phi_x : \mathfrak{S}^*_{m,n} \to \mathfrak{S}_{m,n}$ which deletes the subscripts in the letters of $\tau \in \mathfrak{S}^*_{m,n}$. We omit the details.

3.1 The Upper Bound

To prove the upper bound of Theorem 1.2, essentially the only fact we need is that there are at most n continuously increasing subsequences of a given length k, and as such our proof easily generalizes to a wider set of subsequence problems.

To this end, let $\mathcal{W}$ be a set of words with letters in $[n]$. For $\pi \in \mathfrak{S}_{m,n}$, we define $L_{m,n}(\pi; \mathcal{W})$ to be the maximum length of a word $w \in \mathcal{W}$ which appears as a subsequence in π. For example, if $\mathcal{W}$ consists of every word of the form $i(i+1)\cdots j$ for some $i \le j$, then $L_{m,n}(\pi; \mathcal{W}) = L_{m,n}(\pi)$. We will say that a set of words $\mathcal{W}$ is *prefix closed* if for every $w_1 \cdots w_k \in \mathcal{W}$ we have $w_1 \cdots w_\ell \in \mathcal{W}$ for all $\ell \le k$.

Lemma 3.3. *Let $\mathcal{W}$ be a prefix closed set of words with letters in $[n]$ and let $\mathcal{W}_k \subseteq \mathcal{W}$ be the set of words of length k in $\mathcal{W}$. If $\pi \in \mathfrak{S}_{m,n}$ is chosen uniformly at random, then*

$$\Pr[L_{m,n}(\pi;\mathcal{W}) \geq k] \leq \frac{|\mathcal{W}_k| m^k}{k!}.$$

Proof. For $\tau \in \mathfrak{S}^*_{m,n}$ we define $L^*_{m,n}(\tau;\mathcal{W})$ to be the length of a longest $w \in \mathcal{W}$ such that τ contains a subsequence of type w. By Observation 3.2, it suffices to bound $\Pr[L^*_{m,n}(\tau;\mathcal{W}) \geq k]$ with τ chosen uniformly at random from $\mathfrak{S}^*_{m,n}$.

Because $\mathcal{W}$ is prefix closed, we have $L^*_{m,n}(\tau;\mathcal{W}) \geq k$ if and only if τ contains some subsequence of type $w \in \mathcal{W}_k$, and by definition this happens if and only if τ contains some subsequence of type (w,x) with $w \in \mathcal{W}_k$ and $x \in [m]^k$. For $w \in \mathcal{W}_k$ and $x \in [m]^k$, let $1_{w,x}(\tau)$ be the indicator variable which is 1 if τ contains a subsequence of type (w,x) and which is 0 otherwise. Let $X(\tau) = \sum_{w\in\mathcal{W}_k, x\in[m]^k} 1_{w,x}(\tau)$. By our observations above and Markov's inequality, we find

$$\Pr[L^*_{m,n}(\tau;\mathcal{W}) \geq k] = \Pr[X(\tau) \geq 1] \leq \mathbb{E}[X(\tau)] = \sum_{\substack{w\in\mathcal{W}_k \\ x\in[m]^k}} \Pr[1_{w,x}(\tau) = 1] = |\mathcal{W}_k| m^k \cdot \frac{1}{k!},$$

where the last step used that $1_{w,x}(\tau) = 1$ if and only if the distinct letters $(w_1)_{x_1}, \ldots, (w_k)_{x_k}$ appear in the correct relative order in τ, and this happens with probability $1/k!$. This proves the result. ∎

Proposition 3.4. *Let $\mathcal{W}$ be a prefix closed set of words with letters in $[n]$ and let $\mathcal{W}_k \subseteq \mathcal{W}$ be the words of length k in $\mathcal{W}$. Assume there exists an N such that $|\mathcal{W}_k| \leq N$ for all k. If $\pi \in \mathfrak{S}_{m,n}$ is chosen uniformly at random and N is sufficiently large in terms of m, then*

$$\mathbb{E}[L_{m,n}(\pi;\mathcal{W})] \leq \Gamma^{-1}(N) + O\left(1 + \frac{\log m}{\log(\Gamma^{-1}(N))}\Gamma^{-1}(N)\right).$$

In particular, for $\mathcal{W}$ the set of continuously increasing words $i(i+1)\cdots j$, we have $|\mathcal{W}_k| \leq n$ for all k, so taking $N = n$ gives the upper bound of Theorem 1.2. As another example, if $\mathcal{W}$ is the set of arithmetic progressions, then one can take $N = n^2$ to give an upper bound of roughly $\Gamma^{-1}(n^2)$ for $\mathbb{E}[L_{m,n}(\pi;\mathcal{W})]$. Recent work of Goh and Zhao [GZ20] shows that this bound for arithmetic progressions is tight.

Proof. By using Lemma 3.3 and the trivial bound $\Pr[L_{m,n}(\pi;\mathcal{W}) \geq K] \leq 1$, we find for all integers k that

$$\mathbb{E}[L_{m,n}(\pi;\mathcal{W})] = \sum_{K\geq 1} \Pr[L_{m,n}(\pi;\mathcal{W}) \geq K] \leq k + \sum_{K>k} \frac{Nm^K}{K!}. \tag{13}$$

Let t be the integer such that $(t-1)! < N \leq t!$ and let $k = \left\lceil t + \frac{2\log m}{\log t} t \right\rceil$. Note that this implies $k = t + \frac{C\log m}{\log t} t$ for some $C \geq 2$. Assume N is sufficiently large so that $2m \leq k \leq 2t$. By Lemma 3.1, we have for $K > k \geq t$ that

$$\frac{Nm^K}{K!} \leq \frac{Nm^k}{k!} \cdot \left(\frac{m}{k}\right)^{K-k} \leq \frac{Nm^k}{t! m^{Ct}} \cdot 2^{k-K} \leq 2^{k-K},$$

with this last step using $k \le 2t \le Ct$ and $N \le t!$. Plugging this and our choice of k into (13) gives, after setting $\ell = K - k$,

$$\mathbb{E}[L_{m,n}(\pi;\mathcal{W})] \le \left\lceil t + \frac{2\log m}{\log t} t \right\rceil + \sum_{\ell > 0} 2^{-\ell} \le t + \frac{2\log m}{\log t} t + 2.$$

This gives the desired result since $t < \Gamma^{-1}(N)$. $\square$

3.2 The Lower Bound

For $x, y \in [m]^n$, we define their *Hamming distance* $d_H(x,y) := |\{i \in [n] : x_i \ne y_i\}|$. Our key lemma for proving the lower bound of Theorem 1.2 is the following:

Lemma 3.5. *Let $T \subseteq [m]^n$ be such that any distinct $x, y \in T$ have $d_H(x,y) \ge \delta$ for some integer δ. Then*

$$\Pr[L_{m,n}(\pi) = n] \ge \frac{|T|}{n!}\left(1 - \frac{|T|}{\delta!}\right).$$

Proof. For $\tau \in \mathfrak{S}^*_{m,n}$, let $L^*_{m,n}(\tau)$ denote the length of the longest subsequence of τ of type $i(i+1)\cdots j$. By Observation 3.2, it suffices to prove this lower bound for $\Pr[L^*_{m,n}(\tau) = n]$ where $\tau \in \mathfrak{S}^*_{m,n}$ is chosen uniformly at random. For $x \in [m]^n$, let $A_x(\tau)$ be the event that τ contains a subsequence of type $(12\cdots n, x)$. Observe that

$$\begin{aligned}\Pr[L^*_{m,n}(\tau) = n] &= \Pr\Big[\bigcup_{x \in [m]^n} A_x(\tau)\Big] \ge \Pr\Big[\bigcup_{x \in T} A_x(\tau)\Big] \\ &\ge \sum_{x \in T} \Pr[A_x(\tau)] - \sum_{\substack{x,y \in T \\ x \ne y}} \Pr[A_x(\tau) \cap A_y(\tau)], \qquad (14)\end{aligned}$$

where the last inequality used the Bonferroni inequality (which is essentially a weakening of the principle of inclusion-exclusion); see e.g. [Spe14] for further details on this inequality. To bound (14), we use the following:

Claim 3.6. *If $x, y \in T$ with $x \ne y$, then $\Pr[A_x(\tau)] = 1/n!$ and*

$$\Pr[A_x(\tau) \cap A_y(\tau)] \le \frac{1}{\delta! n!}.$$

Proof. Observe that $A_x(\tau)$ occurs if and only if $1_{x_1}, \ldots, n_{x_n}$ occur in the correct relative order in τ, so $\Pr[A_x(\tau)] = 1/n!$. Let $S = \{i_1 < i_2 < \cdots < i_\delta\}$ be any set of δ indices i such that $y_i \ne x_i$, and note that such a set exists by assumption of T. Let $A^S_y(\tau)$ be the event that $(i_1)_{y_{i_1}} \cdots (i_\delta)_{y_{i_\delta}}$ is a subsequence of τ. Observe that $\Pr[A^S_y(\tau)] = 1/\delta!$ and that this event is independent of $A_x(\tau)$ since these two events concern disjoint sets of letters. Because $A_y(\tau)$ implies $A^S_y(\tau)$, we have

$$\Pr[A_x(\tau) \cap A_y(\tau)] \le \Pr[A_x(\tau) \cap A^S_y(\tau)] = \frac{1}{\delta! n!},$$

proving the result. $\square$

Plugging the results of this claim into (14) and using that the second sum of (14) has at most $|T|^2$ terms gives the desired result. $\square$

The problem of finding $T \subseteq [m]^n$ such that $d_H(x,y) \geq \delta$ with $|T|$ and δ both large is the central problem of coding theory. In particular, a basic greedy argument from coding theory gives the following:

Lemma 3.7. *For any $m \geq 2$ and $1 \leq \delta \leq n/2$, there exists $T \subseteq [m]^n$ such that any two distinct $x, y \in T$ have $d_H(x,y) \geq \delta$ and such that*

$$|T| \geq \frac{m^n}{\delta\binom{n}{\delta}(m-1)^\delta}.$$

Proof. Let $T \subseteq [m]^n$ be a set such that $d_H(x,y) \geq \delta$ for distinct $x, y \in T$ and such that $|T|$ is as large as possible. Let $B(x) = \{y \in [m]^n : d_H(x,y) < \delta\}$, and note that for all x,

$$|B(x)| = \sum_{d=0}^{\delta-1} \binom{n}{d}(m-1)^d \leq \delta\binom{n}{\delta}(m-1)^\delta,$$

with this last step using $\binom{n}{d} \leq \binom{n}{\delta}$ for $d < \delta \leq n/2$. By the maximality of $|T|$, we must have $[m]^n = \bigcup_{x \in T} B(x)$, and thus

$$m^n = \Big|\bigcup_{x \in T} B(x)\Big| \leq |T| \cdot \delta\binom{n}{\delta}(m-1)^\delta,$$

giving the desired bound on $|T|$. $\square$

Combining Lemmas 3.5 and 3.7 gives the following:

Proposition 3.8. *For n sufficiently large in terms of $m \geq 2$, we have*

$$\Pr[L_{m,n}(\pi) = n] \geq \frac{(m/1.03)^n}{2n \cdot n!}.$$

Proof. We start with the following fact.

Claim 3.9. *For all $\epsilon > 0$, there exists a constant $0 < c_\epsilon \leq 1$ such that if $\delta \leq c_\epsilon n$, then $\binom{n}{\delta} \leq (1+\epsilon)^n$.*

Proof. In [Cov99] it is noted that $\binom{n}{\delta} \leq 2^{H(\delta/n)n}$ for all n, δ, where $H(p) := -p\log_2(p) - (1-p)\log_2(1-p)$ is the binary entropy function. Because $H(p)$ tends to 0 as p tends to 0, there exists a constant c such that $2^{H(c)} \leq 1+\epsilon$, and the result follows by taking $c_\epsilon = c$. $\square$

Let $\delta = \frac{2\log m}{\log n}n$, and assume n is sufficiently large in terms of m so that $\delta \leq c_{.01}n$, i.e. so that $\binom{n}{\delta} \leq (1.01)^n$. We also choose n sufficiently large so that $\delta \leq \frac{\log 1.01}{\log m}n$, or equivalently so that $m^\delta \leq (1.01)^n$. Let T be a set as in Lemma 3.7, and by our assumptions above we find

$$|T| \geq \frac{(m/1.03)^n}{n}.$$

Possibly by deleting elements from T we can assume that $|T|$ is exactly the quantity stated above[2], so by Lemma 3.5 it suffices to show $|T|/\delta! \le \frac{1}{2}$. Using the inequality $\delta! \ge (\delta/e)^{\delta}$ and that n is sufficiently large, we have

$$\delta! \ge \exp\left[\delta \cdot (\log(\delta) - 1)\right] \ge \exp\left[\frac{2\log m}{\log n} n \cdot (\log(n) - \log(\log(n)) - 1)\right] \ge \exp\left[\log(m)n\right] = m^n.$$

Thus $|T|/\delta! \le (1.03)^{-n}/n \le \frac{1}{2}$, proving the result. □

With this we can now prove Theorem 1.2.

Proof of Theorem 1.2. The upper bound follows from Proposition 3.4. To prove the lower bound, fix an integer k. For $0 \le j < \lfloor n/k \rfloor$, let $A_j(\pi)$ be the event that π contains the subsequence $(jk+1)(jk+2)\cdots((j+1)k)$.

Claim 3.10. *We have the following:*

(a) *If any $A_j(\pi)$ event occurs, then $L_{m,n}(\pi) \ge k$.*

(b) *The events $A_j(\pi)$ are mutually independent.*

(c) *For all j, we have $\Pr[A_j(\pi)] = \Pr[L_{m,k}(\sigma) = k]$ where $\sigma \in \mathfrak{S}_{m,k}$ is chosen uniformly at random.*

Proof. Part (a) is clear, and (b) follows from the fact that the $A_j(\pi)$ events involve the relative ordering of disjoint sets of letters. For (c), one can consider the map which sends $\pi \in \mathfrak{S}_{m,n}$ to $\sigma \in \mathfrak{S}_{m,k}$ by deleting every letter in π except for $(jk+1), \ldots, ((j+1)k)$ and then relabeling $jk+i$ to i. It is not difficult to see that $A_j(\pi)$ occurs if and only if $L_{m,k}(\sigma) = k$ occurs, and that π being chosen uniformly at random implies σ is chosen uniformly at random. □

Let $p_k = \Pr[L_{m,k}(\sigma) = k]$ and let t be the integer such that $t! \le n < (t+1)!$. The claim above implies that for all k we have

$$\Pr[L_{m,n}(\pi) \ge k] \ge \Pr\left[\bigcup A_j(\pi)\right] = 1 - \Pr\left[\bigcap A_j^c(\pi)\right] = 1 - (1-p_k)^{\lfloor n/k \rfloor} \ge 1 - \exp\left(-\frac{t! p_k}{2k}\right), \quad (15)$$

with this last step using $\lfloor n/k \rfloor \ge n/2k$ for $k \le n$, that $1 - p_k \le e^{-p_k}$, and that $n \ge t!$.

It is easy to see by definition that $p_k \ge 1/k!$ for all m, k; and for n sufficiently large, we have $-e^{-t/2} \ge -t^{-1}$. For such n, by (15) we have for all $k \le t-2$ that

$$\Pr[L_{m,n}(\pi) \ge k] \ge 1 - \exp\left(-\frac{t!}{2k \cdot k!}\right) \ge 1 - \exp\left(-\frac{t}{2}\right) \ge 1 - t^{-1}. \quad (16)$$

[2] Strictly speaking we should take $|T|$ to be the floor of this value to guarantee that it is an integer. This would change our ultimate bound by at most a factor of 2, and this factor of 2 can easily be recovered by sharpening our analysis in various places.

Summing this bound over all $k \leq t-2$ for $m = 1$ gives

$$\mathbb{E}[L_{1,n}(\pi)] \geq \sum_{k \leq t-2} \Pr[L_{1,n}(\pi) \geq k] \geq t - 3.$$

This gives the desired lower bound of $\Gamma^{-1}(n) + \Omega(1)$ for $m = 1$ since $t \geq \Gamma^{-1}(n) - 2$.

We now consider $m \geq 2$. By Proposition 3.8 we have for k sufficiently large in terms of m that $p_k \geq \frac{(m/1.03)^k}{2k \cdot k!}$. Let n be large enough in terms of m so that this bound holds for $k \geq t$. Also let n be large enough so that $\frac{\log m}{\log t} \leq 1$. By Lemma 3.1, if $t \leq k \leq t + \frac{\log m}{100 \log t} t \leq 1.01t$, then $k! \leq t!(1.1)^k m^{t/100}$. Thus by (15) we have

$$\begin{aligned}\Pr[L_{m,n}(\pi) \geq k] &\geq 1 - \exp\left(-\frac{(m/1.03)^k}{4k^2(1.1)^k m^{t/100}}\right) \geq 1 - \exp\left(-\frac{m^{.99t}}{4k^2 \cdot (1.14)^{1.01t}}\right) \\ &\geq 1 - \exp\left(-\frac{(1.7)^t}{8t^2}\right),\end{aligned}$$

where this last step used $m \geq 2$. This quantity is at least $1/2$ for n (and hence t) sufficiently large. Using this together with (16) for $k \leq t-2$ gives, for n sufficiently large in terms of m,

$$\mathbb{E}[L_{m,n}(\pi)] \geq t - 3 + \sum_{t \leq k \leq t + \frac{\log m}{100 \log t} t} \Pr[L_{m,n}(\pi) \geq k] \geq t - 3 + \left(\frac{\log m}{100 \log t} t\right) \cdot \frac{1}{2},$$

proving the desired result. ∎

4 Open Problems and Conjectures

4.1 Longest 1-continuously Increasing Subsequences

In this paper we solved a conjecture of Diaconis, Graham, He, and Spiro [DGHS21] by asymptotically determining $\mathbb{E}[L^1_{m,n}(\pi)]$ provided n is sufficiently large in terms of m. Essentially, we are computing the expected value of the limiting distribution of $L^1_{m,n}(\pi)$ as n goes to infinity, which we shall denote by $L^1_{m,\infty}$. By Observation 1.3, the distribution $L^1_{m,\infty}$ admits an analytic definition as the unique positive integer-valued random variable satisfying

$$\Pr[L^1_{m,\infty} \geq n] = \Pr[L^1_{m,n}(\pi) = n] = \frac{h_m(n)}{|\mathfrak{S}_{m,n}|}, \tag{17}$$

where $h_m(n)$ is as in Theorem 2.1. Our Theorem 1.1 can be restated as a result about the expected value (or the *first* moment) of $L^1_{m,\infty}$.

Using (17), we plot the distribution for $m = 10$ in Figure 2, which suggests that $L^1_{m,\infty}$ seems to approximate a normal distribution of mean roughly $m+1$. This observation is consistent with $\mathbb{E}(L^1_{m,\infty}) \approx m + 1$ implied by Theorem 1.2.

While we cannot estimate the distribution of $L^1_{m,\infty}$ as $m \to \infty$, our techniques for Theorem 1.2 are able to give the exact formula and the $m \to \infty$ asymptotics for the r^{th} moment $\mathbb{E}[(L^1_{m,\infty})^r]$ for any fixed r. The following proposition provides an expression for the r^{th} moment analogous to Theorem 1.1(a).

Proposition 4.1. *For any integer $m \geq 1$, let $\alpha_1, \ldots, \alpha_m$ be the zeroes of $E_m(x) := \sum_{k=0}^{m} \frac{x^k}{k!}$. The r^{th} moment of $L^1_{m,\infty}$ has the exact value*

$$\mathbb{E}[(L^1_{m,\infty})^r] = \sum_{k=1}^{r} \left\{ {r \atop k} \right\} k!\ \Phi_x \left(\left(- \sum_{i=1}^{m} \frac{x}{1-\alpha_i x} \right)^k \right) \Bigg|_{x=-1} \tag{18}$$

where $\left\{ {r \atop k} \right\}$ denotes the Stirling number of the second kind counting the number of set partitions of the set $\{1, \ldots, r\}$ into k blocks, an where Φ_x is the formal Laplace transform with respect to the variable x defined in Section 2.1.

We omit the proof of this result as it uses similar ideas to that of Theorem 1.1(a), along with the basis change identity that transforms the monomial power basis of a polynomial ring $R[y]$ to the falling factorial basis, where the ring R contains the ring of integers $\mathbb{Z}$.

As a corollary of Proposition 4.1, we get a closed form expression for the moment generating function.

Corollary 4.2. *The moment generating function of $L^1_{m,\infty}$ has the following expression:*

$$\sum_{r=1}^{\infty} \mathbb{E}[(L^1_{m,\infty})^r] \frac{z^r}{r!} = \Phi_x \left(\frac{(e^z - 1)t(x)}{1 - (e^z - 1)t(x)} \right) \Bigg|_{x=-1} \tag{19}$$

where $t(x)$ is defined as

$$t(x) := \sum_{i=1}^{m} \frac{x}{1-\alpha_i x}.$$

For any fixed r, we can expand Equation (18) and use ideas similar to those laid out in Section 2.2, to compute the exact value and large m-asymptotics of the r^{th} moment $\mathbb{E}[(L^1_{m,\infty})^r]$. See Table 1 for data on small values of r. We observe the following pattern for the two highest-order terms of $G_r(m)$ based on computational evidence for $r \leq 10$.

r	$\mathbb{E}[(L^1_{m,\infty})^r]$ (Exact)	$G_r(m)$
1	$-S_1(m) - 1$	$m + 1 - \frac{1}{m+2}$
2	$(2m-2)S_2(m) - S_1(m) + 1$	$m^2 + 3m - 2 + \frac{7m^2+25m+28}{(m+2)^2(m+3)}$
3	$(-6m^2 + 15m - 6)S_3(m) + (9m-3)S_2(m) - S_1(m) - 1$	$m^3 + 6m^2 + m + 11 + O(1/m)$
4	$(24m^3 - 104m^2 + 104m - 24)S_4(m) + (-56m^2 + 78m - 12)S_3(m)$	$m^4 + 10m^3 + 15m^2 + 15m + O(1)$
	$+(24m-4)S_2(m) - S_1(m) + 1$	

Table 1: Higher moments of $L^1_{m,\infty}$ (exact values and asymptotics). Here, $S_d(m) := \sum_{i=1}^{m} \alpha_i^{-d} e^{-\alpha_i}$ is the higher exponential sum over all roots α_i of $E_m(x)$, and $G_r(m)$ is the rational function of m that approximates $\mathbb{E}[(L^1_{m,\infty})^r]$ when $m \gg 0$ obtained from (10) together with the exact expression of $\mathbb{E}[(L^1_{m,\infty})^r]$ in terms of exponential sums.

Conjecture 4.3. *For all $r \geq 1$, we have*

$$\mathbb{E}[(L^1_{m,\infty})^r] = m^r + \binom{r+1}{2} m^{r-1} + O(m^{r-2}).$$

We can prove that the r^{th} moment is asymptotic to m^r, but we do not know how to determine the coefficient of m^{r-1}.

We are unable to observe any pattern for the coefficients of lower-order terms of $\mathbb{E}[(L^1_{m,\infty})^r]$. However, a delicate pattern appears to exist when one looks at the r^{th} *centralized* moments of $L^1_{m,\infty}$, namely, the quantity $\mathbb{E}[(L^1_{m,\infty} - \mu)^r]$ with $\mu = \mathbb{E}[L^1_{m,\infty}]$.

Data for the asymptotic values of the centeralized moments is given in Table 2. We note that this data can be viewed as a transformation of the data in Table 1, since the r^{th} centralized moments can be expressed as a linear combination of $0^{\text{th}}, 1^{\text{st}}, \ldots, r^{\text{th}}$ moments. Based on Table 2, we conjecture the following:

r	$H_r(m)$
1	0
2	$m - 1 + \frac{5m^2+14m+13}{(m+2)^2(m+3)}$
3	$m + 1 + O(1/m)$
4	$3m^2 - 5m + 32 + O(1/m)$
5	$10m^2 + m - 149 + O(1/m)$
6	$15m^3 - 20m^2 + 261m - 716 + O(1/m)$
7	$105m^3 - 49m^2 - 1049m + 13819 + O(1/m)$
8	$105m^4 + 70m^3 + 2499m^2 - 8693m - 85877 + O(1/m)$
9	$1260m^4 - 602m^3 - 5592m^2 + 150445m - 350017 + O(1/m)$
10	$945m^5 + 4725m^4 + 27300m^3 - 122769m^2 - 987122m + 18229973 + O(1/m)$

Table 2: The rational function $H_r(m)$ which matches the asymptotics for $\mathbb{E}[(L^1_{m,\infty} - \mu)^r]$. Note that we can compute the $O(1/m)$ terms exactly, but omit writing this due to space limitations.

Conjecture 4.4. *For all $r \geq 2$, we have*

$$\mathbb{E}[(L^1_{m,\infty} - \mu)^r] = c_r m^{\lfloor r/2 \rfloor} + O(m^{\lfloor r/2 \rfloor - 1}),$$

where $\mu = \mathbb{E}[L^1_{m,\infty}]$ and

$$c_r = \begin{cases} \dfrac{r!}{2^{r/2}(r/2)!} = (r-1)!! & r \text{ even,} \\ \dfrac{r!}{3 \cdot 2^{(r-1)/2}((r-3)/2)!} = \dfrac{r-1}{6}\, r!! & r \text{ odd.} \end{cases}$$

We point out that $\mathbb{E}[(L^1_{m,\infty} - \mu)^r]$ (or equivalently, $H_r(m)$ as defined in Table 2) is *a priori* known to be $O(m^r)$ from its expression in terms of moments of $L^1_{m,\infty}$, but the leading degree appears to be $\lfloor r/2 \rfloor$. Thus in particular, this conjecture claims that the $\lceil r/2 \rceil$ highest terms of the expression computing $H_r(m)$ are cancelled, which is essentially a strong assertion about the lower-order terms of $G_r(m)$. The conjecture further asserts the exact leading coefficient c_r of $H_r(m)$, and this surprisingly coincides with the sequence A259877 on the OEIS [Inc15], which arose from a problem in chemistry [AT77].

Our Conjecture 4.4 would imply the asymptotic normality suggested by Figure 2. We note from the datum of $H_2(m)$ that the standard deviation $\sigma = \sqrt{\mathbb{E}[(L^1_{m,\infty} - \mu)^2]}$ of $L^1_{m,\infty}$ is asymptotic to $m^{1/2}$. Thus, this conjecture would imply that the standardized moments $(\frac{L^1_{m,\infty}-\mu}{\sigma})^r$ converge to 0 for r odd and to $\frac{r!}{2^{r/2}(r/2)!}$ for r even. These are exactly the moments of a standard normal distribution, and this fact would imply that $(L^1_{m,\infty} - \mu)/\sigma$ converges in distribution to a standardized normal distribution, see for example [FK16, Corollary 21.8]. It is likely that a certain probabilistic interpretation of Conjecture 4.4 would be the key that explains this combinatorial miracle.

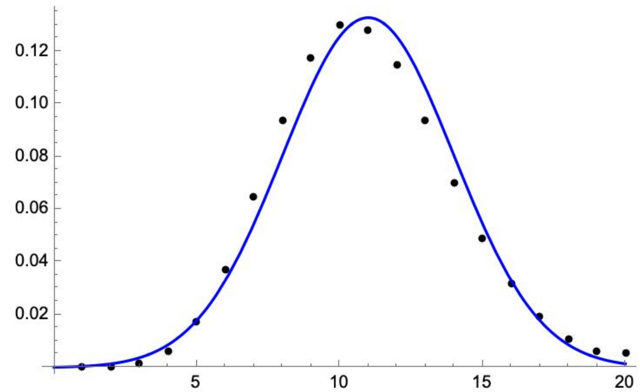

Figure 2: The plots of the probability mass function of $L_{m,\infty}$ (in black) and the probability density function of the normal distribution with mean $m+1$ and variance $m-1$ (in blue), where $m = 10$. Here, the mean and variance are chosen to match the asymptotic mean and variance for $L^1_{m,\infty}$ up to the $O(1)$ term.

4.2 Longest Increasing Subsequences

In this paper, we considered *continuously* increasing subsequences in multiset permutations, and it is natural to consider other types of subsequences in multiset permutations. Perhaps the most natural to consider is the following:

Question 4.5. *For $\pi \in \mathfrak{S}_{m,n}$, let $\widetilde{L}_{m,n}(\pi)$ denote the length of a longest (not necessarily continuously) increasing subsequence in π. What is $\mathbb{E}[\widetilde{L}_{m,n}(\pi)]$ asymptotic to when m is fixed?*

When $m = 1$ it is well known that $\mathbb{E}[\widetilde{L}_{1,n}(\pi)] \sim 2\sqrt{n}$ (see [Rom15]), so Question 4.5 is a natural generalization of this classical problem. See also recent work of Al-Meanazel and Johnson [AMJ20] for some results concerning the distribution of $\widetilde{L}_{m,n}(\pi)$.

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