GENERATING SERIES FOR TORSION-FREE BUNDLES OVER SINGULAR CURVES: RATIONALITY, DUALITY AND MODULARITY

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Abstract. We consider two motivic generating functions defined on a variety, and reveal their tight connection. They essentially count torsion-free bundles and zero-dimensional sheaves.

On a reduced singular curve, we show that the "Quot zeta function" encoding the motives of Quot schemes of 0-dimension quotients of a rank d torsion-free bundle satisfies rationality and a Serre-duality-type functional equation. This is a high-rank generalization of known results on the motivic Hilbert zeta function (the $d = 1$ case) due to Bejleri, Ranganathan, and Vakil (rationality) and Yun (functional equation). Moreover, a stronger rationality result holds in the relative Grothendieck ring of varieties. Our methods involve a novel sublattice parametrization, a certain generalization of affine Grassmannians, and harmonic analysis over local fields.

On a general variety, we prove that the "Cohen–Lenstra zeta function" encoding the motives of stacks of zero-dimensional sheaves can be obtained as a "rank $\to \infty$" limit of the Quot zeta functions.

Combining these techniques, we prove explicit results for the planar singularities associated to the $(2, n)$ torus knots/links, namely, $y^2 = x^n$. The resulting formulas involve summations of Hall polynomials over partitions bounded by a box or a vertical strip, which are of independent interest from the perspectives of skew Cauchy identities for Hall–Littlewood polynomials, Rogers–Ramanujan identities, and modular forms.

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1. Introduction

In this paper, we study enumerative questions about torsion-free bundles and zero-dimensional sheaves over varieties. These objects are of interest in geometric representation theory [24, 25, 42, 43, 49, 63], knot theory [26, 32, 50, 52, 53], classical algebraic geometry and commutative algebra [8, 39, 45, 51, 56], algebraic combinatorics [28–31], quiver varieties [9, 39, 51], mathematical physics and curve counting invariants [3, 33, 54], and arithmetic geometry [7, 36, 38, 63]. More precisely, we prove theorems about the motives and the point counts of the Quot schemes of torsion-free bundles of arbitrary rank and the stacks of coherent sheaves with zero-dimensional support over reduced singular curves, as well as their connection.

1.1. Quot schemes.

Our first goal is to study a *high-rank* generalization of the Hilbert schemes of points. Throughout the paper, let k be a field, and X/k be a reduced curve. For any coherent sheaf $\mathcal{E}$ on X, let $\mathrm{Quot}_{\mathcal{E},n}$ denote the Quot scheme parametrizing quotients of $\mathcal{E}$ with zero-dimensional support and of degree n. When $\mathcal{E}=\mathcal{O}_X$, it is the Hilbert scheme $\mathrm{Hilb}_n(X)$ of n points on X. Let $\pi:\widetilde{X}\to X$ be the normalization of X. Similarly, as a local counterpart, let R be the complete local ring of the germ of a k-curve at a k-point, namely, a reduced complete local k-algebra of Krull dimension 1 with residue field k. For any finitely generated R-module E, let $\mathrm{Quot}_{E,n}$ denote the Quot scheme parametrizing R-submodules of E of k-codimension n. When $E=R$, it is the (punctual) Hilbert scheme of n points on $\operatorname{Spec} R$. Let $R\hookrightarrow\widetilde{R}$ be the normalization of R.

For a k-scheme V of finite type, let $[V]$ denote its class in the Grothendieck ring $K_0(\mathrm{Var}_k)$ of k-varieties. We also refer to $[V]$ as the motive of V. It is the universal measure that satisfies the cut-and-paste property $[V]=[V\setminus Z]+[Z]$ for any closed subscheme $Z\subseteq V$. If $k=\mathbb{C}$, the motive determines the E-polynomial in the sense of mixed Hodge theory ([35]), which further specializes to the virtual weight polynomial and the Euler characteristic. If $k=\mathbb{F}_q$ is a finite field, the motive $[V]$ determines the point count $\#V(\mathbb{F}_q)$. Consider the **Quot zeta functions**

$$Z_{\mathcal{E}}^X(t)=Z_{\mathcal{E}}(t):=\sum_{n\geq 0}[\mathrm{Quot}_{\mathcal{E},n}]\,t^n\in 1+t\cdot K_0(\mathrm{Var}_k)[[t]] \tag{1.1}$$

and

$$Z_E^R(t)=Z_E(t):=\sum_{n\geq 0}[\mathrm{Quot}_{E,n}]\,t^n\in 1+t\cdot K_0(\mathrm{Var}_k)[[t]] \tag{1.2}$$

as formal power series in t with coefficients in $K_0(\mathrm{Var}_k)$. We propose the following general question.

Question 1.1. For $d\geq 1$, what can we say about $Z^X_{\mathcal{O}_X^{\oplus d}}(t)$, $Z^R_{R^{\oplus d}}(t)$, and $Z^R_{\widetilde{R}^{\oplus d}}(t)$?

There are several natural motivations to this question. For example, the case $d=1$ and where R is a plane curve germ corresponds to an important algebro-geometric model for link homologies and (generalizations of) q,t-Catalan numbers that have been substantially investigated in the last decade (see [28–31, 52] and references in [27]). In this direction, a recent Hilb-vs-Quot conjecture [43] relates $Z_R^R(t)$ and $Z_{\widetilde{R}}^R(t)$. It is natural to ask how the conjectural formula generalizes to $d\geq 1$. In another direction, the cohomology of these Quot schemes are of interest in geometric representation theory and curve counting invariants [33, 48, 49, 54], and the Quot zeta function recovers the Betti numbers if a cell decomposition exists.

We now summarize several new results towards Question 1.1. We first make a technical assumption adopted throughout the paper. We do not assume k is algebraically closed for the purpose of point counting over finite fields. However, for simplicity (e.g., to avoid complications from arithmetic), we assume that all singular points of X are k-points, and every singular point of X splits completely with respect to π. Similarly, we assume that as a k-algebra, $\widetilde{R}$ is isomorphic to a finite direct product of the power series ring $k[[T]]$. These assumptions are automatically satisfied when k is algebraically closed. From now on, we will refer to these assumptions by **Assumption** $(*)$.

Let $d \geq 1$ be an integer. We say a coherent sheaf $\mathcal{E}/X$ is a **torsion-free bundle of rank** d if $\mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Frac}(\mathcal{O}_X) \simeq \mathrm{Frac}(\mathcal{O}_X)^{\oplus d}$ and the map $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Frac}(\mathcal{O}_X)$ induced from the natural map $\mathcal{O}_X \hookrightarrow \mathrm{Frac}(\mathcal{O}_X)$ is an injection, where $\mathrm{Frac}(\mathcal{O}_X)$ is the sheaf of rational functions of X. Similarly, we say a finitely generated R-module E is a **torsion-free module of rank** d if $E \hookrightarrow E \otimes_R \mathrm{Frac}(R) \simeq \mathrm{Frac}(R)^{\oplus d}$, where $\mathrm{Frac}(R)$ is the total fraction ring of R. Both $R^{\oplus d}$ and $\widetilde{R}^{\oplus d}$ are examples of torsion-free modules of rank d over R.

Rationality. We are ready to state our first result. Note that we do not assume X or R to be planar.

Theorem 1.2 (Rationality, global)**.** *Assume* $(*)$ *and the notation above. If* $\mathcal{E}$ *is a torsion-free bundle of rank* d *on* X*, then* $Z^X_{\mathcal{E}}(t)$ *is rational in* t*. Moreover,* $Z^X_{\mathcal{E}}(t)/Z^{\widetilde{X}}_{\mathcal{O}^{\oplus d}_{\widetilde{X}}}(t)$ *is a polynomial in* t*.*

This is a common generalization of a recent result of Bejleri, Ranganathan, and Vakil [4] (the $d = 1$ case) and a recent result of Bagnarol, Fantechi, and Perroni [2] (the smooth case). For the latter, when X is a smooth curve, a torsion-free bundle of rank d on X is necessarily a vector bundle of rank d. The rationality then follows from the rationality of the Kapranov motivic zeta function $Z_X(t)$ [41] and their formula

$$Z_{\mathcal{E}}(t) = Z_X(t)Z_X(\mathbb{L}t)\ldots Z_X(\mathbb{L}^{\mathrm{rk}(\mathcal{E})-1}t), \text{ if } X \text{ is smooth and } \mathcal{E}/X \text{ is a vector bundle.} \tag{1.3}$$

Here $\mathbb{L} := [\mathbb{A}^1]$ denotes the Lefschetz motive.

The local counterpart of Theorem 1.2 is as follows.

Theorem 1.3 (Rationality, local)**.** *Assume* $(*)$ *and the notation above. If* E *is a torsion-free module of rank* d *over* R*, then* $Z^R_E(t)$ *is rational in* t*. Moreover,* $Z^R_E(t)/Z^{\widetilde{R}}_{\widetilde{R}^{\oplus d}}(t)$ *is a polynomial in* t*.*

In fact, we can (and *need* to) prove a stronger relative statement, see Theorem 5.38 and Remark 5.36.

Theorem 1.4 (Rationality, local, relative)**.** *If* X *is a quasi-compact* k*-scheme and* $\mathcal{M}$ *is an* X*-family of torsion-free modules of rank* d *over* R[1]*, then the analogous quotient of Quot zeta functions* $Z^R_{\mathcal{M}/X}(t)/Z^{\widetilde{R}}_{\widetilde{\mathcal{M}}/X}(t)$ *is a polynomial with coefficients in* $K_0(\mathbf{Sch}_X)$*, the Grothendieck ring of* X*-schemes.*

By $(*)$, we may assume $\widetilde{R} \simeq k[[T]]^{\times s}$ for some $s \geq 1$; this corresponds to the branching number of the singularity. Using the standard q-Pochhammer symbol

$$(a;q)_n := (1-a)(1-aq)\ldots(1-aq^{n-1}), \qquad (a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots, \tag{1.4}$$

the series $Z^{\widetilde{R}}_{\widetilde{R}^{\oplus d}}(t)$ is given by $1/(t;\mathbb{L})^s_d$ [6]. We define the "numerator part" of $Z^R_E(t)$ by

$$NZ_E(t) := Z_E(t)/Z^{\widetilde{R}}_{\widetilde{R}^{\oplus d}}(t) = (t;\mathbb{L})^s_d Z_E(t) \in K_0(\mathrm{Var}_k)[t]. \tag{1.5}$$

A general formula for $NZ_E(t)$ is given in Remark 5.40. In view of it and our methods in §4–5, the computation of $Z^R_{\widetilde{R}^{\oplus d}}(t)$ seems much more tractable than $Z^R_{R^{\oplus d}}(t)$, see Remark 4.9. This aligns with what is expected in the $d = 1$ case [43].

Functional equation. When R is a planar curve germ, in spite of the paragraph above, $Z^R_{R^{\oplus d}}(t)$ is often preferred because we expect it to satisfy a functional equation. More precisely, we conjecture for a planar singularity R that

$$NZ_{R^{\oplus d}}(t) = (\mathbb{L}^{d^2}t^{2d})^\delta NZ_{R^{\oplus d}}(\mathbb{L}^{-d}t^{-1}), \tag{1.6}$$

where $\delta := \dim_k \widetilde{R}/R$ is the Serre invariant. This is known when $d = 1$ [33, 49, 54], which reflects the Serre duality on the compactified Jacobian scheme parametrizing torsion-free sheaves of rank 1 on X. More generally, R does not have to be planar, but being Gorenstein suffices. We formulate an even more general conjecture as follows. Both the global and the local versions are stated.

[1]By this, we mean an X-family of R-lattices, see Definition 5.1.

Conjecture 1.5 (Functional equation, global)**.** *Assume* $(*)$*, and in addition* X *is projective of arithmetic genus* g_a*. Suppose* $d \geq 0$ *and* ω_X *is a dualizing sheaf of* X*, and let* $\mathcal{E} = \omega_X^{\oplus d}$*. Then we have*

$$Z_{\mathcal{E}}(t) = (\mathbb{L}^{d^2} t^{2d})^{g_a - 1} Z_{\mathcal{E}}(\mathbb{L}^{-d} t^{-1}) \in K_0(\mathrm{Var}_k)(t) \tag{1.7}$$

as rational functions in t*.*

Conjecture 1.6 (Functional equation, local)**.** *Assume* $(*)$*. Suppose* $d \geq 0$ *and* Ω *is the dualizing module of* R*, and let* $E = \Omega^{\oplus d}$*. Then we have*

$$NZ_E(t) = (\mathbb{L}^{d^2} t^{2d})^{\delta} NZ_E(\mathbb{L}^{-d} t^{-1}) \in K_0(\mathrm{Var}_k)(t) \tag{1.8}$$

as rational functions in t*.*

For more about the dualizing module, see §7. If R is Gorenstein (e.g., when R is planar), then we may take $\Omega = R$. As our second major result, we prove that Conjecture 1.6 holds in the sense of point counts over finite fields. Let $|\cdot|_q : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ denote the point-counting measure, so $|\mathbb{L}|_q = q$.

Theorem 1.7. *In the setting of Conjecture 1.6, if* $k = \mathbb{F}_q$*, then*

$$|NZ_E(t)|_q = (q^{d^2} t^{2d})^{\delta} |NZ_E(\mathbb{L}^{-d} t^{-1})|_q \in \mathbb{Q}(t). \tag{1.9}$$

Theorem 1.7 is an arithmetic result about a **lattice zeta function** (in the sense of Solomon [59]), and generalizes a result of Yun [63, Thm. 1.2(2)] (the $d = 1$ case). We note that lattice zeta functions also play a role in representation theory, arithmetic statistics, and subgroup growths [17, 40, 46, 55]. The proof is based on harmonic analysis, cf. [10, 63]. However, "geometrizing" Theorem 1.7 into Conjecture 1.6 remains difficult, see Remark 7.7.

Torus knots/links. We now answer Question 1.1 for the planar singularities corresponding to the $(2, 2m + 1)$ torus knot and the $(2, 2m)$ torus link, where $m \geq 1$. These are among the simplest examples in the $d = 1$ case, but when $d \geq 2$, new ingredients appear in the results. Let $R^{(2,2m+1)}$ be the germ of $y^2 = x^{2m+1}$ and $R^{(2,2m)}$ the germ of $y(y - x^m) = 0$. (When k is not of characteristic two, $R^{(2,2m)}$ is isomorphic to the germ of $y^2 = x^{2m}$.) For an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, let λ'_i denote the size of the i-th column of the Young diagram of λ. Write $\mu \subseteq \lambda$ if the Young diagram of μ is contained in the Young diagram of λ. For partitions λ, μ, ν, let $g^{\lambda}_{\mu\nu}(q) \in \mathbb{Z}[q]$ denote the Hall polynomial; see [47] for an introduction. Finally, let (m^d) denote the partition whose Young diagram is a box with d rows and m columns, and for a partition $\mu \subseteq (m^d)$, let $(m^d) - \mu$ be the partition corresponding to the complement of μ in (m^d).

Theorem 1.8. *Let* $m \geq 1$*,* $d \geq 0$*,* k *be any field, and* $R = R^{(2,2m+1)}$*. Then* $NZ_{R^{\oplus d}}(t) = NZ^R_{\widetilde{R}^{\oplus d}}(t^2)$*. Moreover,* $NZ^R_{\widetilde{R}^{\oplus d}}(t)$ *is an element of* $\mathbb{Z}[t, \mathbb{L}]$ *given by*

$$NZ^R_{\widetilde{R}^{\oplus d}}(t) = \sum_{\mu \subseteq (m^d)} g^{(m^d)}_{\mu, (m^d) - \mu}(\mathbb{L}) \, (\mathbb{L}^d t)^{|\mu|}. \tag{1.10}$$

Remark 1.9. We compare $NZ_{R^{\oplus d}}(t) = NZ^R_{\widetilde{R}^{\oplus d}}(t^2)$ to the Hilb-vs-Quot conjecture in [43]. Let R be any plane curve germ over $\mathbb{C}$, and assume that both $NZ_R(t)$ and $NZ^R_{\widetilde{R}}(t)$ are polynomials in $\mathbb{L}$ and t. Since the virtual weight polynomial of $\mathbb{A}^1$ is given by $\chi(\mathbb{A}^1, t) = t^2$, [43, Conj. 1] is equivalent to

$$NZ_R(t) = NZ^R_{\widetilde{R}}(t)|_{\mathbb{L} \mapsto \mathbb{L}t}, \tag{1.11}$$

a change of variable *different* from $t \mapsto t^2$. When $R = R^{(2,2m+1)}$, we have $NZ^R_{\widetilde{R}}(t) = \sum_{i=0}^{m} (\mathbb{L}t)^i$, so $NZ^R_{\widetilde{R}}(t^2)$ and $NZ^R_{\widetilde{R}}(t)|_{\mathbb{L} \mapsto \mathbb{L}t}$ coincide. However, when $d \geq 2$, our formula implies $NZ^R_{\widetilde{R}^{\oplus d}}(t^2) \neq NZ^R_{\widetilde{R}^{\oplus d}}(t)|_{\mathbb{L} \mapsto \mathbb{L}t}$.

Theorem 1.10. *Let $m \geq 1$, $d \geq 0$, k be any field, and $R = R^{(2,2m)}$. Then $NZ^R_{R^{\oplus d}}(t)$ and $NZ^R_{\widetilde{R}^{\oplus d}}(t)$ are elements of $\mathbb{Z}[t, \mathbb{L}]$ given by*

$$NZ_{R^{\oplus d}}(t) = \sum_{\lambda,\mu,\nu \subseteq (m^d)} g^{(m^d)}_{\lambda,(m^d)-\lambda}(\mathbb{L})\, g^{\lambda}_{\mu\nu}(\mathbb{L})\, (t;\mathbb{L})^2_{d-\lambda'_m} t^{|\lambda|} (\mathbb{L}^d t)^{|\lambda|-|\mu|} \frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\lambda'_m}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\mu'_m}} \tag{1.12}$$

and

$$NZ^R_{\widetilde{R}^{\oplus d}}(t) = \sum_{\mu \subseteq (m^d)} g^{(m^d)}_{\mu,(m^d)-\mu}(\mathbb{L})\, (\mathbb{L}^d t)^{|\mu|} \frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_d}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{d-\mu'_1}}. \tag{1.13}$$

Theorem 1.10 brings more questions than answers. First, no apparent change of variable seems to relate $NZ^R_{\widetilde{R}^d}(t)$ and $NZ_{R^d}(t)$, but they seem similar enough that they might have a common refinement by introducing an additional variable; see §9.7. A natural question is to identify the refinement, possibly as an extra homological grading, which would shed light to the formulation of a $d \geq 2$ analogue of the Hilb-vs-Quot conjecture. Second, Theorem 1.7 and (1.12) implies a combinatorial identity stating that $f_{d,m}(t, \mathbb{L})$ defined as the right-hand side of (1.12) satisfies $f_{d,m}(t, q) = (q^{d^2} t^{2d})^m f_{d,m}(q^{-d} t^{-1}, q)$. This can be viewed as a bounded analogue of the skew Cauchy identity, see Remark 9.4, and has yet no direct proof for $m \geq 2$. Seeking a combinatorial interpretation for this identity would be of interest. Numerical data in §9.7 suggest that $f_{d,m}(-t, q)$ should have positive and unimodal coefficients.

Remark 1.11. On the positive side, the specializations of $NZ_{R^d}(t)$ and $NZ_{\widetilde{R}^d}(t)$ at $t = 1$ or $\mathbb{L} = 1$ (namely, taking the Euler characteristic when $k = \mathbb{C}$) do have predictable behaviors, see §9.4. Finally, formulas in Theorems 1.8 and 1.10 can be written explicitly in terms of q-Pochhammer symbols by applying (3.5) and Theorem 3.4. Moreover, for the $(2, 2)$ link, (1.12) can be simplified, see §9.6.

1.2. **Motivic Cohen–Lenstra zeta function.** In [36], from the perspective of matrix Diophantine equations, the first author studied an enumerative invariant attached to any variety that counts zero-dimensional coherent sheaves. It generalizes a zeta function introduced by Cohen and Lenstra in the seminal work [14] and is closely related to a wide range of topics in number theory, algebraic geometry, representation theory, and mathematical physics, such as matrix Diophantine equations, commuting varieties, quiver representations, and Donaldson–Thomas theory [3, 7, 9, 21, 38].

More precisely, for an arbitrary field k, let $K_0(\mathrm{Stck}_k)$ be the Grothendieck ring of k-stacks with affine stabilizers, which is isomorphic to the localization of $K_0(\mathrm{Stck}_k)$ at $[\mathrm{GL}_n]$ for all $n \geq 1$ [20]. We canonically map $K_0(\mathrm{Stck}_k)$ into $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$, the completion of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ in the dimension filtration, as in [9, §2]. For any scheme X/k of finite type, let $\mathrm{Coh}_n(X)$ be the stack of 0-dimension coherent sheaves on X of degree n. As a local analogue, for a complete local k-algebra R of finite type with residue field k, let $\mathrm{Coh}_n(R)$ be the stack of R-modules of k-dimension n. Define power series

$$\widehat{Z}_X(t) := \sum_{n \geq 0} [\mathrm{Coh}_n(X)]\, t^n \in 1 + t \cdot K_0(\mathrm{Stck}_k)[[t]] \tag{1.14}$$

and

$$\widehat{Z}_R(t) := \sum_{n \geq 0} [\mathrm{Coh}_n(R)]\, t^n \in 1 + t \cdot K_0(\mathrm{Stck}_k)[[t]] \tag{1.15}$$

with coefficients in $K_0(\mathrm{Stck}_k)$. We call them the **motivic Cohen–Lenstra zeta function**. When R is the germ of a curve satisfying $(*)$ in §1.1, we define the "numerator part" of $\widehat{Z}_R(t)$ by

$$\widehat{NZ}_R(t) = \widehat{Z}_R(t)/\widehat{Z}_{\widetilde{R}}(t), \tag{1.16}$$

where $\widetilde{R}$ is the normalization of R. Note that $\widehat{Z}_{\widetilde{R}}(t) = 1/(\mathbb{L}^{-1}t; \mathbb{L}^{-1})^s_\infty$ in $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}, t]]$, where $\widetilde{R} \simeq k[[T]]^{\times s}$. (We also treat $(\mathbb{L}^{-1}t; \mathbb{L}^{-1})_\infty$ as an element in $K_0(\mathrm{Stck}_k)[[t]]$ via Euler's identity.)

There are philosophical reasons to expect the lack of a satisfactory theory for $\widehat{Z}_R(t)$ in general: the category of finite-dimensional R-module is wild, our moduli problem has no framing (or stability condition) in the sense of quiver varieties, etc. Indeed, little has been known beyond the cases where

R is the germ of a smooth curve or smooth surface [36]. However, we show that $\widehat{Z}_R(t)$ and $Z_{R^{\oplus d}}(t)$ defined in §1.1 are intimately connected, as below.

Let R be a (not necessarily reduced or one-dimensional) complete local k-algebra of finite type with residue field k, and let $\mathfrak{m}$ denote the maximal ideal of R. We refer to this by **Assumption** $(**)$, which is much weaker than $(*)$. We equip $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]][[t]]$ (denoted $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1},t]]$ in short) with the topology induced t-coefficientwise from the topology on $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$ given by the dimension filtration. Note that it restricts to the usual topology on the power series ring $\mathbb{Z}[[\mathbb{L}^{-1},t]]$ via the canonical inclusion $\mathbb{Z}[[\mathbb{L}^{-1},t]] \hookrightarrow K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1},t]]$.

Theorem 1.12 (Rank $\to\infty$ limit)**.** *Let R satisfy $(**)$. Then $\lim_{d\to\infty} Z_{R^{\oplus d}}(\mathbb{L}^{-d}t)$ converges, and*

$$\widehat{Z}_R(t) = \lim_{d\to\infty} Z_{R^{\oplus d}}(\mathbb{L}^{-d}t). \tag{1.17}$$

We now focus on the planar singularities $R^{(2,n)}$. Using the new techniques granted by Theorem 1.12, we compute $\widehat{Z}_{R^{(2,n)}}(t)$ for $n\geq 2$. For an integer partition λ, define a polynomial in q by

$$a_q(\lambda) := q^{\sum_{i\geq 1}\lambda_i'^2} \prod_{i\geq 1} (q^{-1};q^{-1})_{\lambda_i'-\lambda_{i+1}'}. \tag{1.18}$$

The polynomial $a_q(\lambda)$ arises in counting automorphisms, see [47, p. 181].

Theorem 1.13. *For $m\geq 1$, we have*

$$\widehat{NZ}_{R^{(2,2m+1)}}(t) = \sum_{\mu:\mu_1\leq m} \frac{t^{2|\mu|}}{a_{\mathbb{L}}(\mu)} \in \mathbb{Z}[[\mathbb{L}^{-1},t]], \tag{1.19}$$

where the sum extends over all partition μ whose parts are at most m.

Theorem 1.14. *For $m\geq 1$, we have*

$$\widehat{NZ}_{R^{(2,2m)}}(t) = (\mathbb{L}^{-1}t;\mathbb{L}^{-1})_\infty^2 \sum_{\substack{\lambda,\mu,\nu\\ \lambda_1\leq m}} \frac{g^{\lambda}_{\mu\nu}(\mathbb{L})\,(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\lambda_m'}\, t^{2|\lambda|-|\mu|}}{a_{\mathbb{L}}(\lambda)(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\mu_m'}(\mathbb{L}^{-1}t;\mathbb{L}^{-1})^2_{\lambda_m'}} \in \mathbb{Z}[[\mathbb{L}^{-1},t]]. \tag{1.20}$$

Remark. For $k=\mathbb{F}_q$ and $n\geq 2$, the formulas imply $|\widehat{NZ}_{R^{(2,n)}}(t)|_q$ converges for $t\in\mathbb{C}$, see Remark 9.5.

Example 1.15. Using Theorem 1.13 and Theorem 1.14 and techniques from [36], one can count solutions to certain matrix equations over finite fields. For example, (1.19) with $m=1$ implies

$$\left[\{(A,B)\in \mathrm{Mat}_n(k)^2 : AB=BA, A^2=B^3\}\right] = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor} (-1)^j \mathbb{L}^{\frac{1}{2}(3j^2-j)+n(n-2j)} \frac{(\mathbb{L};\mathbb{L})_n}{(\mathbb{L};\mathbb{L})_j(\mathbb{L};\mathbb{L})_{n-2j}}, \tag{1.21}$$

and setting $\mathbb{L}\mapsto q$ gives the count over $k=\mathbb{F}_q$. Analogous counts with nilpotent matrices can be similarly obtained.

Special values and Rogers–Ramanujan. The celebrated Rogers–Ramanujan (RR) identities are two formal identities of the form "infinite sum=infinite product=modular form". We propose a surprising arithmetic geometric framework to obtain infinite families of new generalizations of RR identities. Namely, we take the special values $\widehat{NZ}_R(\pm 1)$. The case of $R^{(2,2m+1)}$ makes the connection evident: by (1.19), one of Andrews–Gordon's generalized RR identities [1, Cor. 7.8] can be restated as

$$\widehat{NZ}_{R^{(2,2m+1)}}(\pm 1)|_{\mathbb{L}^{-1}\mapsto q} = \sum_{\mu:\mu_1\leq m} \frac{1}{a_{q^{-1}}(\mu)} = \prod_{n\not\equiv 0,\pm(m+1) \pmod{2m+3}} (1-q^n)^{-1} \in \mathbb{Z}[[q]], \tag{1.22}$$

On the other hand, considering $R^{(2,2m)}$ yields and suggests new infinite families of identities.

Theorem 1.16. *For $m\geq 1$, we have $\widehat{NZ}_{R^{(2,2m)}}(1)=1$.*

Conjecture 1.17. *For $m \geq 1$, we have a formal identity*

$$\widehat{NZ}_{R^{(2,2m)}}(-1)|_{\mathbb{L}^{-1}\mapsto q} = \frac{(q^2;q^2)_\infty (q^{m+1};q^{m+1})_\infty^2}{(q;q)_\infty^2 (q^{2m+2};q^{2m+2})_\infty} \in \mathbb{Z}[[q]]. \tag{1.23}$$

Remark. When $m = 1$, both identities were previously obtained by the first author [36].

An important feature of these identities is that the "sum side" involves partitions with parts bounded by m. This, and the presence of Hall(–Littlewood) polynomials, notably align with an influential framework due to Griffin, Ono, and Warnaar [34] from a completely different perspective, namely, affine Kac–Moody algebras. However, both Theorem 1.16 and Conjecture 1.17 seem new. In view of the Oblomkov–Rasmussen–Shende conjecture [52], our framework might be connected to knot theory. To conclude, we summarize our conjectures.

Conjecture 1.18. *Let R be a plane curve germ over $\mathbb{C}$. Then*

(a) *There exists a formal power series $H_R(t;q) \in \mathbb{Z}[[q,t]]$ such that $\widehat{NZ}_R(t) = H_R(t;\mathbb{L}^{-1})$.*
(b) ([36]) *For complex numbers t, q with $|q| < 1$, the series $H_R(t;q)$ converges.*
(c) *There is a rational number $\kappa(R)$ such that $q^{\kappa(R)} H_R(1;q)$ and $q^{\kappa(R)} H_R(-1;q)$ are Fourier expansions of modular functions (of weight 0).*
(d) *$H_R(t;q)$ (or at least $H_R(\pm 1;q)$) is determined by the topology of the associated link of R.*

Remark 1.19. The case $R = R^{(2,n)}$ provably verifies Conjecture 1.18(a)(b). Partially conditional on Conjecture 1.17, Conjecture 1.18(c) is verified by $R = R^{(2,n)}$ with $\kappa(R^{(2,2m+1)}) = -m/(24m+36)$ (see for example [58]) and $\kappa(R^{(2,2m)}) = 0$.

1.3. Methods and organization.

To obtain our results, we need a mixture of many techniques. We first overview the methods revolving the Quot zeta functions. Some initial steps are standard: in §2 we clarify why every global statement can be reduced to local statements, and in the opening discussions of §4 we recognize $Z_E(t)$ for torsion-free E as instances of (motivic) lattice zeta functions. After that, we use two different techniques to compute the lattice zeta functions. The proofs of Theorems 1.3, 1.8, and 1.10 use novel ingredients: an explicit parametrization of sublattices based on the extension operator $L \mapsto \widetilde{R}L$ (see §4), and a generalization of affine Grassmannians (see §5). On the other hand, the proof of Theorem 1.7 is a straightforward application of Tate's thesis in the same manner as Bushnell–Reiner [10] and Yun [63], see 7. These two methods seem mutually irreplaceable in the theorems they prove respectively.

To relate the motivic Cohen–Lenstra zeta function to the Quot zeta functions, we "overparametrize" $\mathrm{Coh}_n(R)$ with $\mathrm{Quot}_{R^{\oplus d},n}$ in the spirit of ADHM constructions (cf. [39, 51]), and show that it has a well-controlled overcount. Geometrizing this idea gives Theorem 6.1, see §6. The rank $\to \infty$ limit formula (Theorem 1.12) then follows directly.

Finally, in explicit computations for the $y^2 = x^n$ singularities, Hall polynomials arise naturally. This is essentially because the local ring in question satisfies a special property that its quotient ring by the conductor ideal is isomorphic to a DVR quotient, see §8.2. To simplify expressions involving Hall polynomials, we use standard transformation identities of q-hypergeometric series, see §9.

The paper is organized as follows. In §2 we show Theorem 1.3 implies Theorem 1.2 and Conjecture 1.6 implies Conjecture 1.5. In §3 we state some preliminaries. In §4 we prove the rationality (Theorem 1.3) up to point counts. In §5 we make it motivic. In §6 we prove the rank $\to \infty$ limit formula (Theorem 1.12). In §7 we prove the functional equation (Theorem 1.7). In §8 we apply the method in §4 to analyze the case of $R^{(2,n)}$. In §9 we prove some formal identities arising from the explicit computations above, and give further observations and data.

We remark that most of the sections can be read independently, since they employ different techniques, except that §5 and §8 require §4.

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2. Reduction to the local case

The following lemma relates the global and local Quot zeta functions. Recall the notation in (1.3).

Lemma 2.1. *Suppose X/k is a reduced curve, $x_1, \dots, x_r$ are k-points of X, and $U := X \setminus \{x_1, \dots, x_r\}$ is smooth. Let $\mathcal{E}$ be a torsion-free bundle on X of rank d. For $1 \le i \le r$, let $R_i := \widehat{\mathcal{O}}_{X,x_i}$ and let E_i be the completed stalk of $\mathcal{E}$ at x_i. Then*

$$Z_{\mathcal{E}}(t) = Z_U(t) Z_U(\mathbb{L}t) \dots Z_U(\mathbb{L}^{d-1}t) \prod_{i=1}^{r} Z_{E_i}^{R_i}(t). \tag{2.1}$$

Proof. For $n', n_1, \dots, n_r \ge 0$, let $n = n' + \sum_{i=1}^r n_i$ and consider the locally closed subscheme $Q_{n',(n_1,\dots,n_r)}$ of $\mathrm{Quot}_{\mathcal{E},n}$ parametrizing subsheaves of $\mathcal{E}$ whose quotient has length n' support in U and length n_i support at x_i. We have

$$\mathrm{Quot}^X_{\mathcal{E},n} = \bigsqcup_{n'+\sum_{i=1}^r n_i = n} Q_{n',(n_1,\dots,n_r)} = \bigsqcup_{n'+\sum_{i=1}^r n_i = n} \mathrm{Quot}^U_{\mathcal{E}|_U, n'} \times \prod_{i=1}^{r} \mathrm{Quot}^{R_i}_{E_i, n_i}, \tag{2.2}$$

where the superscripts are introduced to disambiguate. It follows from the definition of the series that $Z^X_{\mathcal{E}}(t) = Z^U_{\mathcal{E}|_U}(t) \prod_{i=1}^r Z^{R_i}_{E_i}(t)$. Since U is a smooth curve, $\mathcal{E}|_U$ is a vector bundle of rank d. Applying (1.3) to U and $\mathcal{E}|_U$, the proof is complete. □

From here, assuming ($*$), a standard cut-and-paste argument shows that Theorem 1.3 implies Theorem 1.2 and Conjecture 1.6 implies Conjecture 1.5.

3. Preliminaries

3.1. **Some commutative algebra.** The following lemmas will be repetitively used later. Here, R, A are commutative rings with 1.

Lemma 3.1. *Let R be a direct product of finitely many Noetherian local rings, and M be a finitely generated module over R. Then $\mathrm{GL}_n(R)$ acts on $\mathrm{Surj}(R^n, M)$ transitively, namely, if F_1, F_2 are surjections $R^n \twoheadrightarrow M$, then there is $g \in \mathrm{GL}_n(R)$ such that $F_2 = F_1 \circ g$.*

Proof. Without loss of generality, we may assume R is a Noetherian local ring. The conclusion then follows from the uniqueness of the minimal free resolution, see [19, Thm. 20.2]. □

Lemma 3.2. *Let A be a ring and V_1, V be A-modules with $V_1 \subseteq V$. If V_1 is a direct summand of V, then for any decomposition $V = V_1 \oplus V_2$, there is a bijection*

$$\begin{gathered}\{W \subseteq_A V : W + V_1 = V\} \to \{(W', \varphi) : W' \subseteq_A V_1, \varphi \in \mathrm{Hom}_A(V_2, V_1/W')\}, \\ W \mapsto (W \cap V_1, \varphi_W),\end{gathered} \tag{3.1}$$

where φ_W is defined by the composition $V_2 \to V \to V/W \simeq V_1/(W \cap V_1)$.

Proof. It is routine to check that the following construction gives the inverse map: given (W', φ), define W to be the kernel of the map $1 \oplus \varphi : V_1 \oplus V_2 \to V_1/W'$, where 1 denotes the quotient map $V_1 \to V_1/W'$. □

3.2. **Hall polynomials.** We recall the definition of Hall polynomials, which appear in Theorems 1.8, 1.10, and 1.14. Let $(R, \pi, \mathbb{F}_q)$ be a discrete valuation ring (DVR) with uniformizer π and residue field $\mathbb{F}_q$. For a finite(-cardinality) R-module M, the **type** of M is the unique partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with

$$M \simeq \bigoplus_{i=1}^{l} R/\pi^{\lambda_i} R, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l. \tag{3.2}$$

We denote the type of M by $\lambda_R(M)$, or simply $\lambda(M)$.

For finite R-modules $N \subseteq M$, we define the **cotype** of N in M to be $\lambda(M/N)$. The definition and basic properties of Hall polynomials are given below, which are all we need in §8.

Definition-Theorem 3.3 ([47])**.** Given partitions λ, μ, and ν, there exists a universal polynomial $g^{\lambda}_{\mu\nu}(q) \in \mathbb{Z}[q]$ such that whenever q is a prime power and R is any DVR with residue field $\mathbb{F}_q$, the value $g^{\lambda}_{\mu\nu}(q)$ represents the number of submodules with type μ and cotype ν of a fixed R-module with type λ. Moreover, we have (i) $g^{\lambda}_{\mu\nu}(t) = g^{\lambda}_{\nu\mu}(t)$, and (ii) $g^{\lambda}_{\mu\nu}(t) = 0$ unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \subseteq \lambda$.

In §9, we will need the following explicit formulas. Define the q-binomial coefficients by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_r (q;q)_{n-r}} \in \mathbb{Z}[q], \qquad n \geq r \geq 0. \tag{3.3}$$

Theorem 3.4 ([62])**.** *For partitions $\mu \subseteq \lambda$, the following identity in $\mathbb{Z}[q]$ holds:*

$$g^{\lambda}_{\mu}(q) := \sum_{\nu} g^{\lambda}_{\mu\nu}(q) = q^{\sum_{i\geq 1} \mu'_i(\lambda'_i - \mu'_i)} \prod_{i \geq 1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_{q^{-1}}. \tag{3.4}$$

Note that $g^{\lambda}_{\mu}(q)$ can be interpreted as the number of submodules with type (or cotype) μ of a fixed R-module with type λ.

Remark 3.5. As an important special case, let $\lambda = (m^d)$ be the $d \times m$ box, and $\mu \subseteq (m^d)$, where $d, m \geq 0$. We have that $g^{(m^d)}_{\mu\nu}(q)$ is nonzero for precisely one partition ν, namely, $\nu = (m^d) - \mu$. It follows that

$$g^{(m^d)}_{\mu,(m^d)-\mu}(q) = g^{(m^d)}_{\mu}(q) = \frac{q^{d|\mu|}}{q^{\sum_{i\geq 1} \mu_i'^2} \prod_{i \geq 1} (q^{-1};q^{-1})_{\mu'_i - \mu'_{i+1}}} \frac{(q^{-1};q^{-1})_d}{(q^{-1};q^{-1})_{d-\mu'_1}}. \tag{3.5}$$

We note that $g^{(m^d)}_{\mu}(q)$ does not depend on m, as long as $m \geq \lambda_1$.

In the proof of Proposition 8.9, we need the following consequence of Nakayama's lemma.

Lemma 3.6. *Let $(R, \pi, \mathbb{F}_q)$ be a DVR, $d \geq 0$, and M be an R-module of type μ with $\mu'_1 \leq d$. Then the number of surjective homomorphisms $R^d \twoheadrightarrow M$ is $q^{d|\mu|} \dfrac{(q^{-1};q^{-1})_d}{(q^{-1};q^{-1})_{d-\mu'_1}}$.*

Proof. Note that $M/\pi M \simeq \mathbb{F}_q^{\mu'_1}$. By Nakayama's lemma, the probability that a homomorphism $R^d \to M$ be surjective is the same as the probability that an $\mathbb{F}_q$-linear map $\mathbb{F}_q^d \to \mathbb{F}_q^{\mu'_1}$ be surjective. □

4. Parametrizing sublattices

The crux of the proof of Theorems 1.3, 1.8, and 1.10 is a new and somewhat more elementary method to compute the lattice zeta function, which is the goal of this section. We first review the lattice zeta function of Solomon [59] and clarify its connection with $Z_E(t)$.

Assume R satisfies $(*)$, recall the notation therein, and let $K = \mathrm{Frac}(R)$ be the total fraction ring of R. For a free module V over K of finite rank, a (full) **R-lattice** in V is a finitely generated R-submodule L of K^d such that $KL = V$. For $d \geq 0$, an R-lattice in $K^d := K^{\oplus d}$ is also called an R-lattice of rank d. When k is a finite field, the lattice zeta function of an R-lattice M is defined by

$$\zeta_M^R(z) = \zeta_M(z) := \sum_{L \subseteq_R M} |M/L|^{-z}, \tag{4.1}$$

where the sum extends over all R-sublattices of M. It is closely related to the Quot zeta function of M as an R-module: we have $\zeta_M(z) = |Z_M(q^{-z})|_q$. In this sense, even when k is infinite, the Quot zeta function of an R-lattice M can also be called the **motivic lattice zeta function** of M.

If E is a torsion-free module over R of rank d, we may find an R-lattice of rank d that is isomorphic to E by fixing an isomorphism $E \otimes_R K \simeq K^d$ and considering the embedding $E \hookrightarrow E \otimes_R K \simeq K^d$. Denote its image by M, then $Z_E(t) = Z_M(t)$, and thus is an instance of the motivic lattice zeta function. As usual, even though $Z_E(t)$ depends only on the isomorphism class of E, it is necessary that we work with a fixed embedding $E \hookrightarrow K^d$.

The lattice zeta function has been intensively studied [10–12, 44, 63] and is of interest in number theory. The usual method is to divide the set of sublattices of M into isomorphism classes, and express the contribution of each isomorphism class as an integral on $\mathrm{GL}_d(K)$, see §7. However, torsion-free modules of rank $d \geq 2$ are hard to classify. Moreover, the family is "wild" in both categorical and geometric senses [15, 16], even for simple examples such as the $y^2 = x^3$ singularity. In order to make explicit computations and prove geometric statements about $Z_E(t)$, we need a parametrization of sublattices of M that does not depend on the classification of torsion-free modules up to isomorphism. In this section, we describe our parametrization in general, which will be applied to the case of $y^2 = x^n$ singularities in §8. Along the way, we keep track of various lattice counts over finite fields, which lead to the point-counting versions of Theorems 1.3.

4.1. **Notation.** Let k, R be given, and assume $(*)$ and the notation of §1.1. Fix an isomorphism $\widetilde{R} \simeq k[[T_1]] \times \dots k[[T_s]]$; we call $s \geq 1$ the **branching number** of R. For $1 \leq i \leq s$, denote $\widetilde{R}_i = k[[T_i]]$ and $K_i = k((T_i))$. Let $e_i \in \widetilde{R}$ be the idempotent associated to the factor $k[[T_i]]$, and let $T = (T_1, \dots, T_s) \in \widetilde{R}$. By abuse of notation, define $T_i := e_i T$ as an element of $\widetilde{R}$, which identify $k[[T_i]]$ with a subring of $\widetilde{R}$.

For any k-vector spaces $L_1 \subseteq L_2$, define the (additive) **index** of L_1 in L_2 to be $[L_2 : L_1] := \dim_k L_2/L_1$. In particular, this applies to a pair of nested R-lattices $L_1 \subseteq L_2$ of K^d. The definition of index can be uniquely extended to any pair of R-lattices by $[L_1 : L_2] + [L_2 : L_3] + [L_3 : L_1] = 0$.

Let $\mathfrak{c}$ be the **conductor** of R, namely, the largest ideal of $\widetilde{R}$ such that $\mathfrak{c} \subseteq R$. For $1 \leq i \leq s$, let c_i be such that $\mathfrak{c} = (T_i^{c_i})_{i=1}^s \widetilde{R}$, and write $c := \sum_{i=1}^s c_i$. We have $[\widetilde{R} : \mathfrak{c}] = c$.

From now on, we fix $d \geq 1$, and unless otherwise specified, a lattice refers to a lattice in K^d.

4.2. **Extension fibers.** Our main idea to parametrize R-lattices L is to first consider $\widetilde{R}L$ as an $\widetilde{R}$-lattice in K^d. Parametrizations of $\widetilde{R}$-lattices are well-known, for example, using the Hermite normal form [59]. For each $\widetilde{R}$-lattice $\widetilde{L}$, we shall classify R-lattices L whose extension is $\widetilde{L}$. We define the **extension fiber** by

$$E_R(\widetilde{L}) := \{L \subseteq_R \widetilde{L} : \widetilde{R}L = \widetilde{L}\}. \tag{4.2}$$

The following simple observation is crucial.

Lemma 4.1. *For all $L \in E_R(\widetilde{L})$, we have $\mathfrak{c}\widetilde{L} \subseteq L \subseteq \widetilde{L}$.*

Proof. This is because $L = RL \supseteq \mathfrak{c}L = (\mathfrak{c}\widetilde{R})L = \mathfrak{c}(\widetilde{R}L) = \mathfrak{c}\widetilde{L}$. □

Remark 4.2. It follows that $[\widetilde{L} : L] \leq cd$ for $L \in E_R(\widetilde{L})$.

Recall that our precise goal is to parametrize R-sublattices of a given R-lattice M. To this end, we define the **restricted extension fiber** by

$$E_R(\widetilde{L};M) := \{L \in E_R(\widetilde{L}) : L \subseteq M\}, \tag{4.3}$$

and study $E_R(\widetilde{L};M)$ as $\widetilde{L}$ varies. In order for $E_R(\widetilde{L};M)$ to be nonempty, it is clearly necessary that $\widetilde{L} \subseteq \widetilde{R}M$. However, $\widetilde{L}$ can have arbitrarily high index in $\widetilde{R}M$. To restrict our attention to $\widetilde{L}$ of bounded indices in $\widetilde{R}M$, we prove the following "padding" lemma.

Lemma 4.3 (Padding lemma)**.** *Let $\underset{\sim}{M}$ be an $\widetilde{R}$-lattice contained in an R-lattice M. Then for any $\widetilde{R}$-lattice $\widetilde{N}$, there exists $g \in \mathrm{GL}_d(K)$ with $g(\widetilde{N}) = \widetilde{N} + \underset{\sim}{M}$ such that $N \mapsto g \cdot N$ induces a bijection $E_R(\widetilde{N};M) \to E_R(\widetilde{N}+\underset{\sim}{M};M)$. In fact, any $g \in \mathrm{Isom}_{\widetilde{R}}(\widetilde{N}, \widetilde{N}+\underset{\sim}{M})$ such that $gx - x \in \underset{\sim}{M}$ does the job, and such g exists.*

Proof. Denote $\widetilde{L} := \widetilde{N} + \underset{\sim}{M}$. Then the $\widetilde{R}$-linear map $\widetilde{N} \to \widetilde{L}/\underset{\sim}{M}$ induced by the inclusion map $\widetilde{N} \to \widetilde{L}$ is a surjection. By comparing the above surjection with the quotient map $\widetilde{L} \to \widetilde{L}/\underset{\sim}{M}$, noting that $\widetilde{N} \simeq \widetilde{L} \simeq \widetilde{R}^d$, and applying Lemma 3.1, we conclude that there exists $g \in \mathrm{Isom}_{\widetilde{R}}(\widetilde{N}, \widetilde{L})$ such that $gx \equiv x \bmod \underset{\sim}{M}$ for all $x \in \widetilde{N}$.

Now let such g be given. Then g induces an isomorphism $E_R(\widetilde{N}) \to E_R(\widetilde{L})$. It remains to show that $gN \subseteq M$ for any $N \in E_R(\widetilde{N};M)$ and $g^{-1}L \subseteq M$ for any $L \in E_R(\widetilde{L};M)$. To prove the first claim, we recall that $\underset{\sim}{M} \subseteq M$ and note for any $N \in E_R(\widetilde{N};M)$ and $x \in N$ that

$$gx \equiv x \equiv 0 \pmod{M}. \tag{4.4}$$

The proof of the second claim is similar, where we note that $g^{-1}y \equiv y \bmod \underset{\sim}{M}$ for all $y \in \widetilde{L}$. □

Thanks to the padding lemma, it suffices to consider $E_R(\widetilde{L};M)$ for $\widetilde{L} \supseteq \underset{\sim}{M}$. Given such $\widetilde{L}$, we still need to classify all $\widetilde{N}$ such that $\widetilde{N} + \underset{\sim}{M} = \widetilde{L}$. This is achieved by the following lemma.

Lemma 4.4. *Let $\underset{\sim}{M} \subseteq \widetilde{L}$ be two $\widetilde{R}$-lattices. Define $r_i = \mathrm{rk}_i(\widetilde{L}/\underset{\sim}{M}) := \mathrm{rk}_{\widetilde{R}_i}(\widetilde{L}/\underset{\sim}{M} \otimes_{\widetilde{R}} \widetilde{R}_i)$, the minimal number of generators of $\widetilde{L}/\underset{\sim}{M} \otimes_{\widetilde{R}} \widetilde{R}_i$ as an $\widetilde{R}_i$-module. Then there is a bijection*

$$\left\{\widetilde{N} \subseteq_{\widetilde{R}} \widetilde{L} : \widetilde{N} + \underset{\sim}{M} = \widetilde{L}\right\} \to \prod_{i=1}^{s} \left\{(\widetilde{N}_i', \varphi_i) : \widetilde{N}_i' \subseteq_{\widetilde{R}_i} \widetilde{R}_i^{d-r_i}, \varphi_i \in \mathrm{Hom}_{\widetilde{R}_i}(\widetilde{R}_i^{r_i}, \widetilde{R}_i^{d-r_i}/\widetilde{N}_i')\right\}. \tag{4.5}$$

Moreover, under this bijection, we have $[\widetilde{L} : \widetilde{N}] = \sum_{i=1}^{s} [\widetilde{R}_i^{d-r_i} : \widetilde{N}_i']$.

Proof. Without loss of generality, we may assume $s = 1$ and drop the subscript i throughout. In light of Lemma 3.2, it suffices to find a direct sum decomposition $\widetilde{L} = L^{\mathrm{in}} \oplus L^{\mathrm{out}}$, such that $L^{\mathrm{in}} \simeq \widetilde{R}^{d-r}$, $L^{\mathrm{out}} \simeq \widetilde{R}^r$, and

$$\left\{\widetilde{N} \subseteq_{\widetilde{R}} \widetilde{L} : \widetilde{N} + \underset{\sim}{M} = \widetilde{L}\right\} = \left\{\widetilde{N} \subseteq_{\widetilde{R}} \widetilde{L} : \widetilde{N} + L^{\mathrm{in}} = \widetilde{L}\right\}. \tag{4.6}$$

(The notation L^{in} and L^{out} will soon become clear. In fact, it will turn out that $L^{\mathrm{in}} \subseteq \underset{\sim}{M}$ and L^{out} can be viewed as "the part of $\widetilde{L}$ outside $\underset{\sim}{M}$" in some sense. We drop the tilde in the notation as a warning that L^{in} and L^{out} are not lattices of rank d.)

To construct L^{in} and L^{out}, we consider the quotient map $\pi : \widetilde{L} \to \widetilde{L}/\underset{\sim}{M}$ and fix any surjection $\alpha : \widetilde{R}^r \to \widetilde{L}/\underset{\sim}{M}$, where r is the minimal number of generators of $\widetilde{L}/\underset{\sim}{M}$. Since $\widetilde{L}$ is free, thus projective as a module, there exists a map $\beta : \widetilde{L} \to \widetilde{R}^r$ such that $\pi = \alpha \circ \beta$. Since α is an isomorphism mod $\mathfrak{m}$ and π is surjective, by Nakayama's lemma, β is surjective. Let $L^{\mathrm{in}} := \ker(\beta)$. To define L^{out}, note that β has a splitting $\gamma : \widetilde{R}^r \to \widetilde{L}$ because the target of β is free. Let $L^{\mathrm{out}} = \mathrm{im}(\gamma)$. Then it is clear that $L^{\mathrm{in}} \simeq \widetilde{R}^{d-r}$, $L^{\mathrm{out}} \simeq \widetilde{R}^r$, and $\widetilde{L} = L^{\mathrm{in}} \oplus L^{\mathrm{out}}$.

It remains to prove (4.6). We note that $\widetilde{N} + \underset{\sim}{M} = \widetilde{L}$ if and only if the composition $\widetilde{N} \hookrightarrow \widetilde{L} \to \widetilde{L}/\underset{\sim}{M}$ is surjective. Similarly, $\widetilde{N} + L^{\mathrm{in}} = \widetilde{L}$ if and only if the composition $\widetilde{N} \hookrightarrow \widetilde{L} \to \widetilde{L}/L^{\mathrm{in}} = \widetilde{R}^r$ is surjective.

But these two maps differ by α, which is an isomorphism mod $\mathfrak{m}$, so their surjectivities are equivalent by Nakayama's lemma. □

Remark 4.5. When $\underset{\sim}{M}$ and $\widetilde{L}$ are fixed, our construction realizes (4.5) as an isomorphism of k-varieties. Moreover, on a stratum of the right-hand side with $[\widetilde{R}_i^{d-r_i} : \widetilde{N}_i'] = n_i$, forgetting all φ_i for $1 \le i \le s$ induces a fibration onto $\prod_{i=1}^s \mathrm{Quot}^{\widetilde{R}_i}_{\widetilde{R}_i^{d-r_i},n_i}$ with fiber $\mathbb{A}^{\sum_{i=1}^n r_i n_i}$. This will be needed in §5.

Remark 4.6. We note that the space of φ_i's, namely $\mathrm{Hom}_{\widetilde{R}_i}(\widetilde{R}_i^{r_i}, \widetilde{R}_i^{d-r_i}/\widetilde{N}_i')$, is a k-vector space of dimension $r_i[\widetilde{R}_i^{d-r_i} : \widetilde{N}_i']$. When $k = \mathbb{F}_q$ is a finite field, by Solomon's formula [59]

$$\sum_{M \subseteq_{\mathbb{F}_q[[T]]} \mathbb{F}_q[[T]]^d} t^{[\mathbb{F}_q[[T]]^d : M]} = \frac{1}{(t;q)_d}, \tag{4.7}$$

we have

$$\sum_{\widetilde{N} : \widetilde{N} + \underset{\sim}{M} = \widetilde{L}} t^{[\widetilde{L}:\widetilde{N}]} = \prod_{i=1}^s \sum_{\widetilde{N}_i' \subseteq_{\widetilde{R}_i} \widetilde{R}_i^{d-r_i}} q^{r_i[\widetilde{R}_i^{d-r_i}:\widetilde{N}_i']} t^{[\widetilde{R}_i^{d-r_i}:\widetilde{N}_i']} \tag{4.8}$$

$$= \prod_{i=1}^s \frac{1}{(q^{r_i}t;q)_{d-r_i}} = \frac{\prod_{i=1}^s (t;q)_{r_i}}{(t;q)_d^s}. \tag{4.9}$$

We summarize the recipe to parametrize sublattices of M while keeping track of their indices. First, we need to choose and fix $\underset{\sim}{M}$ as in Lemma 4.3. For example, we may choose $\underset{\sim}{M}$ to be $\mathfrak{c}M$, or the unique maximal $\widetilde{R}$-lattice contained in M. To account for all possible $E_R(\widetilde{L}; M)$ we need to consider, define the set of **boundary $\widetilde{R}$-lattices** for $M/\underset{\sim}{M}$ to be

$$\widetilde{\partial}_{\underset{\sim}{M}}(M) := \{\widetilde{L} \supseteq_{\widetilde{R}} \underset{\sim}{M} : E_R(\widetilde{L}; M) \neq \varnothing\}. \tag{4.10}$$

Assuming the bijections in Lemmas 4.3 and 4.4 are fixed in advance, giving $N \subseteq_R M$ amounts to

- Choosing $\widetilde{L} \in \widetilde{\partial}_{\underset{\sim}{M}}(M)$;
- Choosing $\widetilde{N}$ using the data at the right-hand side of (4.5), and independently choose $L \in E_R(\widetilde{L}; M)$.

The target lattice N is then obtained from L via the bijection in Lemma 4.3. Moreover, the index $[M : N]$ can be read from the classification datum by

$$[M : N] = [M : \widetilde{L}] + [\widetilde{L} : \widetilde{N}] + [\widetilde{N} : N] = [M : \widetilde{L}] + \left(\sum_{i=1}^s [\widetilde{R}_i^{d-r_i} : \widetilde{N}_i']\right) + [\widetilde{L} : L]. \tag{4.11}$$

The classification above immediately leads to the following formula for $|NZ_M(t)|_q$ when $k = \mathbb{F}_q$ is a finite field. It implies the point-counting version of Theorem 1.3. In §8, we will perform explicit computations based on Proposition 4.7.

Proposition 4.7. *Assume the previous notation, and assume $k = \mathbb{F}_q$. Given an R-lattice M, and a choice of $\widetilde{R}$-lattice $\underset{\sim}{M}$ with $\underset{\sim}{M} \subseteq M$, we have*

$$|NZ_M(t)|_q := (t;q)_d^s \sum_{N \subseteq_R M} t^{[M:N]} = \sum_{\widetilde{L} \in \widetilde{\partial}_{\underset{\sim}{M}}(M)} \left(\prod_{i=1}^s (t;q)_{\mathrm{rk}_i(\widetilde{L}/\underset{\sim}{M})}\right) \sum_{L \in E_R(\widetilde{L};M)} t^{[M:L]}. \tag{4.12}$$

Proof. The formula follows from combining Lemma 4.1, Lemma 4.3, Remark 4.6, and (4.7). □

Corollary 4.8. *In the setting above, $|NZ_M(t)|_q$ is a polynomial in t of degree at most $(2c + s)d$.*

Proof. We choose $\underset{\sim}{M} = \mathfrak{c}M$ when applying (4.12). The factor $\prod_{i=1}^{s}(t;q)_{\mathrm{rk}_i(\widetilde{L}/\underset{\sim}{M})}$ is a polynomial in t of degree at most sd. To bound $[M:L]$ for $L \in E_R(\widetilde{L};M)$, we note that $[M:L] = [M:\widetilde{L}] + [\widetilde{L}:L]$, where

$$[M:\widetilde{L}] \leq [M:\underset{\sim}{M}] = [M:\mathfrak{c}M] \leq [\widetilde{R}M:\mathfrak{c}\widetilde{R}M] = cd, \tag{4.13}$$

and $[\widetilde{L}:L] \leq cd$ by Remark 4.2. It follows that $|NZ_M(t)|_q$ is a polynomial in t of degree at most $(2c+s)d$. □

Remark 4.9. If M is an $\widetilde{R}$-lattice $\widetilde{M}$ (i.e., $M \simeq \widetilde{R}^d$), then choosing $\underset{\sim}{M} = \widetilde{M}$ simplifies (4.12) to

$$|NZ^R_{\widetilde{M}}(t)|_q = \sum_{L \in E_R(\widetilde{M})} t^{[\widetilde{M}:L]}. \tag{4.14}$$

Remark 4.10. The polynomial $|NZ^R_M(t)|_q$ has an easy special value at $t=1$. In (4.12), noting that $(1;q)_n = 0$ for $n > 0$, the only $\widetilde{L}$ that contributes to $|NZ^R_M(1)|_q$ is $\widetilde{L} = \underset{\sim}{M}$. Therefore, we have

$$|NZ^R_M(1)|_q = \#E_R(\underset{\sim}{M}) = \#E_R(\widetilde{R}^d) \tag{4.15}$$

for any R-lattice M of rank d. In particular, the special value of $|NZ^R_M(t)|_q$ at $t=1$ depends only on R and d but not on M. When $d=1$, this compares to the $\widetilde{J}_R(0)$ statement of [63, Thm. 2.5(2)].

So far, we have proved the point-count version of Theorem 1.3. However, this argument not yet implies the motivic statements since our parametrization involves noncanonical bijections to be fixed in Lemmas 4.3 and Lemmas 4.4. We resolve this gap next.

5. Geometry of lattices

In this section we introduce a generalization of affine Grassmannians and study their properties. These will be used to prove the motivic rationality. Assume the setting and notation in §4.1 and fix R throughout this section.

5.1. **Affine Grassmannians and their generalizations.** We first introduce the moduli space of all R-lattices, which is a generalization of the notion of the affine Grassmannian. Let S be a k-algebra. We write $S\widehat{\otimes}R$ *resp.* $S\widehat{\otimes}K$ as the completion of $S \otimes R$ *resp.* $S \otimes K$ with respect to the $1 \otimes \mathfrak{m}$-adic topology. For example, $S\widehat{\otimes}k[[T]] = S[[T]]$ and $S\widehat{\otimes}k((T)) = S((T))$.

Definition 5.1. An S-family of rank d R-lattices is a finitely generated $S\widehat{\otimes}R$-submodule $\mathcal{L} \subseteq S\widehat{\otimes}K^d$, such that $\mathcal{L} \otimes (S\widehat{\otimes}K) = S\widehat{\otimes}K^d$ and $S\widehat{\otimes}K^d/\mathcal{L}$ is locally free over S. An S-family of rank d R-lattices can be thought of as a family of torsion-free R-bundles over S. Let $\mathrm{Gr}_{R,d}$ be the functor over $\mathbf{Alg}_k$ sending S to S-families of rank d R-lattices. We often abbreviate $\mathrm{Gr}_{R,d}$ as Gr_R, while keeping d implicit.

Remark 5.2. If $\widetilde{R} = k[[T_1]] \times \cdots \times k[[T_s]]$, we write $\underline{T} = T_1 + \cdots + T_s$ in place of T for clarity. Since $\mathcal{L}$ is finitely generated, one can show using the conductor that there exists $i > 0$ such that $\underline{T}^i S\widehat{\otimes}\widetilde{R} \subseteq \mathcal{L} \subseteq \underline{T}^{-i} S\widehat{\otimes}\widetilde{R}$.

Remark 5.3. Given a finitely generated locally free $S\widehat{\otimes}R$-module $\mathcal{L}$, there exists a Zariski cover $S' \to S$ such that $\mathcal{L}_{S'}$ is free over $S'\widehat{\otimes}R$. To see this, one can assume that S is a local ring. Then $S\widehat{\otimes}R$ is also a local ring. Therefore, $\mathcal{L}_{S'}$ is free.

Proposition 5.4. Gr_R *is represented by an ind-projective-scheme*[2] *over* k.

[2]Recall that an ind-scheme X is a functor, which can be written as a union of subfunctors $X = \bigcup_i X_i$, such that each X_i is represented by a scheme, and $X_i \hookrightarrow X_{i+1}$ is a closed immersion. If $\mathcal{P}$ is a property of schemes, e.g., projective. Then an ind-scheme is said to be ind-$\mathcal{P}$, if every X_i has $\mathcal{P}$.

Proof. The proof is routine. Let $\mathbb{G}r_{\widetilde{R},i}$ be the scheme $\bigsqcup_{r\geq 0}\mathrm{Gr}(r, \underline{T}^{-i}\widetilde{R}/\underline{T}^{i}\widetilde{R})$. Let $\mathrm{Gr}_{R,i} : \mathbf{Alg}_k \to \mathbf{Set}$ be the subfunctor of $\mathbb{G}r_{\widetilde{R},i}$, defined as

$$\mathrm{Gr}_{R,i} : S \mapsto \{S\widehat{\otimes}R\text{-module quotients } \underline{T}^{-i}S\widehat{\otimes}\widetilde{R}/\underline{T}^{i}S\widehat{\otimes}\widetilde{R} \to Q \text{ that are locally free over } S\}. \tag{5.1}$$

By Remark 5.2, S-families of rank d R-lattices $\mathcal{L}$ such $\underline{T}^{i}S\widehat{\otimes}\widetilde{R} \subseteq \mathcal{L} \subseteq \underline{T}^{-i}S\widehat{\otimes}\widetilde{R}$ correspond bijectively to S-points of $\mathrm{Gr}_{R,i}$. Therefore $\mathrm{Gr}_R = \bigcup_i \mathrm{Gr}_{R,i}$. If we pick a finite k-spanning set $\{g_1, g_2, \ldots, g_n\}$ of R modulo $\underline{T}^{i}R$, we can re-interpret $\mathrm{Gr}_{R,i}$ as

$$\mathrm{Gr}_{R,i} : S \mapsto \begin{Bmatrix} S\text{-submodules } M \subseteq \underline{T}^{-i}S\widehat{\otimes}\widetilde{R}/\underline{T}^{i}S\widehat{\otimes}\widetilde{R} \text{ with quotients locally free over } S, \\ \text{and } g_jM \subseteq M \text{ for } j = 1, 2, \ldots, n. \end{Bmatrix}. \tag{5.2}$$

Since each g_i imposes a closed condition, we find that $\mathrm{Gr}_{R,i}$ is a closed subscheme of $\mathbb{G}r_{\widetilde{R},i}$. In particular, $\mathrm{Gr}_{R,i}$ is projective. This implies that $\mathrm{Gr}_{R,i} \hookrightarrow \mathrm{Gr}_{R,i+1}$ is a closed embedding. Therefore Gr_R is ind-projective. □

Corollary 5.5. *Let S be a k-algebra. An S-families of R-lattices is defined over a finite type subalgebra.*

Proof. An S-family of R-lattices is an S-point of Gr_R, which factors through some $\mathrm{Gr}_{R,i}$ for $i \gg 0$. Since $\mathrm{Gr}_{R,i}$ is finite type, it commutes with direct limit. Writing S as a direct union of finite type subalgebras, we see that an S-point is factors though some finite type subalgebra of S. □.

Remark. From now on, we write $\mathcal{U}$ for the universal lattice over Gr_R. We will also use the notation $\widetilde{\mathcal{U}}$ for the universal lattice over $\mathrm{Gr}_{\widetilde{R}}$.

5.1.1. *Bounded affine Grassmannians.* Now we work relatively over a certain base scheme. It is harmless to assume that the base is affine. So let S be any affine algebra. Let $\mathcal{M}, \mathcal{M}'$ be S-families of R-lattices, and let S' be any S-algebra. Write $\mathrm{Gr}_{R,S}$ for the base change of Gr_R to $\operatorname{Spec} S$. Define three subfunctors of $\mathrm{Gr}_{R,S} : \mathbf{Alg}_S \to \mathbf{Set}$ as follows:

(a) $\mathrm{Gr}_R([\mathcal{M}, \infty))$: its S' points are S'-families of lattices that contain $\mathcal{M}_{S'}$.
(b) $\mathrm{Gr}_R((-\infty, \mathcal{M}])$: its S' points are S'-families of lattices that are contained in $\mathcal{M}_{S'}$.
(c) $\mathrm{Gr}_R([\mathcal{M}', \mathcal{M}]) = \mathrm{Gr}_R([\mathcal{M}', \infty)) \cap \mathrm{Gr}_R((-\infty, \mathcal{M}])$.

To ease notation, we will also denote $\mathrm{Gr}_R((-\infty, \mathcal{M}])$ *resp.* $\mathrm{Gr}_R([\mathcal{M}', \mathcal{M}])$ as $\mathrm{Gr}_R(\mathcal{M})$ *resp.* $\mathrm{Gr}_R(\mathcal{M}', \mathcal{M})$. The readers are encouraged to assume $S = k$ upon first reading.

Lemma 5.6. $\mathrm{Gr}_R([\mathcal{M}, \infty))$ *and* $\mathrm{Gr}_R(\mathcal{M})$ *are ind-closed in* $\mathrm{Gr}_{R,S}$*, while* $\mathrm{Gr}_R(\mathcal{M}', \mathcal{M})$ *is a closed subscheme of* $\mathrm{Gr}_{R,S}$*.*

Proof. Recall the subfunctor $\mathrm{Gr}_{R,i}$ from Proposition 5.4. As noted in Remark 5.2, one can pick i sufficiently large so that $\underline{T}^{i}S\widehat{\otimes}\widetilde{R} \subseteq \mathcal{M} \subseteq \underline{T}^{-i}S\widehat{\otimes}\widetilde{R}$. Then $\mathrm{Gr}_R(\mathcal{M}) \cap \mathrm{Gr}_{R,i}$ *resp.* $\mathrm{Gr}_R([\mathcal{M}, \infty)) \cap \mathrm{Gr}_{R,i}$ is just the pullback of the closed subfunctor of $\mathbb{G}r_{\widetilde{R},i,S}$ whose S'-points are locally free S'-quotients of $\underline{T}^{-i}S'\widehat{\otimes}\widetilde{R}/\underline{T}^{i}S'\widehat{\otimes}\widetilde{R}$ whose kernels are contained in $\mathcal{M}/\underline{T}^{i}S'\widehat{\otimes}\widetilde{R}$ *resp.* contain $\mathcal{M}/\underline{T}^{i}S'\widehat{\otimes}\widetilde{R}$. Therefore, $\mathrm{Gr}_R(\mathcal{M})$ and $\mathrm{Gr}_R([\mathcal{M}, \infty))$ are ind-closed. The fact that $\mathrm{Gr}_R(\mathcal{M}', \mathcal{M})$ is a closed subscheme follows easily. □

Let p_1 *resp.* p_2 be the projection of $\mathrm{Gr}_{R,S}$ to Gr_R *resp.* $\operatorname{Spec} S$. We have pullback lattices $p_2^*\mathcal{M}$ and $p_1^*\mathcal{U}$ over $\mathrm{Gr}_{R,S}$. By the definition of families of lattices, the quotient $(p_2^*\mathcal{M}/p_1^*\mathcal{U})_{\mathrm{Gr}_R(\mathcal{M})}$ is a locally free $\mathrm{Gr}_{R,S}$-module. Accordingly, $\mathrm{Gr}_R(\mathcal{M})$ admits the following grading

$$\mathrm{Gr}_R(\mathcal{M}) = \bigsqcup_{n\geq 0} \mathrm{Gr}_R^n(\mathcal{M}), \tag{5.3}$$

where each component $\mathrm{Gr}_R^n(\mathcal{M})$ is an open and closed subscheme of $\mathrm{Gr}_R(\mathcal{M})$, and $(p_2^*\mathcal{M}/p_1^*\mathcal{U})_{\mathrm{Gr}_R^n(\mathcal{M})}$ is locally free of rank n. By taking $\mathrm{Gr}_R^n(\mathcal{M}', \mathcal{M}) = \mathrm{Gr}_R(\mathcal{M}', \mathcal{M}) \cap \mathrm{Gr}_R^n(\mathcal{M})$ we also obtain a grading of $\mathrm{Gr}_R(\mathcal{M}', \mathcal{M})$. But note that $\mathrm{Gr}_R^n(\mathcal{M}', \mathcal{M}) = \emptyset$ if $n \gg 0$. A similar grading also exists for $\mathrm{Gr}_R([\mathcal{M}, \infty))$, but won't be used. One reason to study these bounded affine Grassmannians is their relation to Quot schemes. Let the relative Quot scheme $\mathrm{Quot}^R_{\mathcal{M}/S,n}$ be the functor $\mathbf{Alg}_S \to \mathbf{Set}$ sending S'/S to $S'\widehat{\otimes}R$-quotients (equivalently, $S'\otimes R$-quotients) of $\mathcal{M}_{S'}$ which are locally free S'-modules of rank n.

Proposition 5.7 (Bounded Grassmannian ⇔ Quot)**.** $\mathrm{Gr}^n_R(\mathcal{M}) \simeq \mathrm{Quot}^R_{\mathcal{M}/S,n}$.

Proof. Let S'/S be an object of Alg_S. We crucially note that since $S'\widehat{\otimes}K^d/\mathcal{M}_{S'}$ is S'-flat, for any finitely generated $S'\widehat{\otimes}R$-submodule $\mathcal{L}\subseteq \mathcal{M}_{S'}$, we have that $S'\widehat{\otimes}K^d/\mathcal{L}$ is S'-flat if and only if $\mathcal{M}_{S'}/\mathcal{L}$ is S'-flat, see [60, 00HM]. It is then routine to check that $\mathcal{L}\mapsto (j:\mathcal{M}_{S'}\twoheadrightarrow \mathcal{M}_{S'}/\mathcal{L})$ and $\ker(j)\leftarrow j$ gives a bijection $\mathrm{Gr}^n_R(\mathcal{M})(S')\to \mathrm{Quot}^R_{\mathcal{M}/S,n}(S')$. □

5.1.2. *Padding cells.* Let $\mathcal{M}'\subseteq\mathcal{M}$ be an inclusion of two S-families of R-lattices. Define a subfunctor $\mathrm{Pd}_R(\mathcal{M}',\mathcal{M})\subseteq \mathrm{Gr}_R(\mathcal{M})$ whose value over a ring extension S'/S are S'-families of sublattices $\mathcal{L}\subseteq\mathcal{M}_{S'}$ such that $\mathcal{L}+\mathcal{M}'_{S'}=\mathcal{M}_{S'}$. Note that $\mathrm{Pd}_R(\mathcal{M}',\mathcal{M})$ comes with a distinguished S-point corresponding to $\mathcal{M}$. We call this point $\underline{e}$. Furthermore, $\mathrm{Pd}_R(\mathcal{M}',\mathcal{M})$ is equipped with a grading $\mathrm{Pd}^n_R(\mathcal{M}',\mathcal{M})$ that comes from intersecting with $\mathrm{Gr}^n_R(\mathcal{M})$.

Lemma 5.8. $\mathrm{Pd}_R(\mathcal{M}',\mathcal{M})\subseteq \mathrm{Gr}_R(\mathcal{M})$ *is ind-open in* $\mathrm{Gr}_R(\mathcal{M})$.

Proof. There exists i_0, such that $\mathcal{M}'\supseteq \underline{T}^{i_0}S\widehat{\otimes}\widetilde{R}$. Define, for $i\geq i_0$, $\mathbb{G}r'_i$ to be the S-relative Grassmannian parametrizing locally free quotients of the bundle $\mathcal{M}/\underline{T}^iS\widehat{\otimes}\widetilde{R}$. Then $\mathrm{Gr}_R(\mathcal{M})$ is the direct union of the closed subschemes $\mathrm{Gr}_R(\mathcal{M})\cap\mathbb{G}r'_i$. Then $\mathrm{Pd}_{\widetilde{R}}(\mathcal{M}',\mathcal{M})\cap\mathbb{G}r'_i$ is just the pullback to $\mathrm{Gr}_R(\mathcal{M})\cap\mathbb{G}r'_i$ of the open subfunctor of $\mathbb{G}r'_i$ whose S'-points are locally free S'-quotients of $\mathcal{M}/\underline{T}^iS'\widehat{\otimes}\widetilde{R}$ whose kernel $\mathcal{K}\subseteq\mathcal{M}/\underline{T}^iS'\widehat{\otimes}\widetilde{R}$ satisfies the property that $\mathcal{K}+(\mathcal{M}'/\underline{T}^iS'\widehat{\otimes}\widetilde{R})=\mathcal{M}/\underline{T}^iS'\widehat{\otimes}\widetilde{R}$. □

Remark 5.9. Padding cells geometrize of the notions in Lemma 4.4 and the two remarks that follow. Let $\underset{\sim}{M}\subseteq\widetilde{M}$ be two R-lattices over k, and define $r_i=\mathrm{rk}_i(\widetilde{M}/\underset{\sim}{M})$ as in Lemma 4.4. The computation in Remark 4.6 translates to the fact that, as elements in $K_0(\mathrm{Var}_k)[[t]]$,

$$\sum_{n\geq 0}[\mathrm{Pd}^n_R(\underset{\sim}{M},\widetilde{M})]t^n=(t;\mathbb{L})_d^{-s}\prod_{i=1}^{s}(t;\mathbb{L})_{r_i}. \tag{5.4}$$

5.1.3. *Local triviality.* In the following we work exclusively over $R=\widetilde{R}$. Let S be a k-algebra and let $\underset{\sim}{\mathcal{M}}$, $\widetilde{\mathcal{M}}$ be S-families of $\widetilde{R}$ lattices such that $\underset{\sim}{\mathcal{M}}\subseteq\widetilde{\mathcal{M}}$. We will call $\widetilde{\mathcal{M}}$ **trivial**, if it is free. Equivalently, if it is isomorphic to the base change of a lattice $\widetilde{M}$ over k along the morphism $k\to S$. Similarly, the inclusion $\underset{\sim}{\mathcal{M}}\subseteq\widetilde{\mathcal{M}}$ is called **trivial**, if it is isomorphic to the base change of an inclusion of lattices $\underset{\sim}{M}\subseteq\widetilde{M}$ over k along the structural morphism. Clearly, in both cases, $\widetilde{M}$ can be taken as $\widetilde{R}^d$.

Lemma 5.10 (Zariski local triviality of $\widetilde{\mathcal{M}}$)**.** *Any S-family of $\widetilde{R}$-lattices is Zariski locally trivial.*

Proof. This is an easy consequence of [64, Lemma 1.8], in which one first reduces to the Noetherian case by Corollary 5.5, and invokes the local criterion of flatness. □

Lemma 5.11 (Constructible local triviality of $\underset{\sim}{\mathcal{M}}\subseteq\widetilde{\mathcal{M}}$)**.** *There is a stratification of S such that $\underset{\sim}{\mathcal{M}}\subseteq\widetilde{\mathcal{M}}$ is trivial over each stratum.*

Proof. By Corollary 5.5, we can reduce to the case where S is finite type. We can then reduce to the case where S is integral and the branching number of R is $s=1$. By Noetherian induction, it suffices to show that after shrinking S, $\underset{\sim}{\mathcal{M}}\subseteq\widetilde{\mathcal{M}}$ is trivial. From Lemma 5.10, we may also assume that $\widetilde{\mathcal{M}}$ is already trivial, i.e., $\widetilde{\mathcal{M}}=\bigoplus_{j=1}^d S[[T]]e_j$.

Let F be the fraction field of S. The theory of Smith normal forms implies that there is a basis $\{v_j\}_{1\leq j\leq d}$ of $\widetilde{\mathcal{M}}_F$ such that $\{T^{n_j}v_j\}_{1\leq j\leq d}$ is a basis of $\underset{\sim}{\mathcal{M}}_F$. For each $1\leq j\leq d$, we can write

$$v_j=\sum_{l=1}^{d}\sum_{r\geq 0}a_{jlr}T^re_l,\ a_{jlr}\in F. \tag{5.5}$$

Let $i \geq 1$ be an integer such that $T^i \widetilde{M} \subseteq \underset{\sim}{M}$ (this is the case if and only if $n_j \leq i$ for all j). Then the set $\{a_{jlr}\}_{1\leq j,l\leq d, 0\leq r\leq i}$ is nonzero only for finitely many elements. So possibly shrinking S, we can assume that $\{a_{jlr}\}_{1\leq j,l\leq d, 0\leq r\leq i} \subseteq S$. It is then easy to see that there exists $\{v'_j\}_{1\leq j\leq d} \subseteq \widetilde{M}$, with the property $v_j \equiv v'_j \mod T^i$.

Note that $(a_{jl0}) \in \mathrm{GL}_d(F)$. Possibly shrinking S, we can assume $(a_{jl0}) \in \mathrm{GL}_d(S)$. Then $\{v'_j\}_{1\leq j\leq d}$ is also a basis of $\widetilde{M}$. On the other hand, let $\underset{\sim}{M}'$ be the sublattice of $\widetilde{M}$ spanned by the set $\{T^{n_j} v'_j\}_{1\leq j\leq d}$. It is clear that $T^i S[[T]]$ is contained in both $\underset{\sim}{M}$ and $\underset{\sim}{M}'$. Moreover, $\underset{\sim}{M}/T^i\widetilde{M} = \underset{\sim}{M}'/T^i\widetilde{M}$ as submodules of $\widetilde{M}/T^i\widetilde{M}$. It follows that $\underset{\sim}{M} = \underset{\sim}{M}'$. This implies that $\{T^{n_j} v'_j\}_{1\leq j\leq d}$ is a basis of $\underset{\sim}{M}$. This shows that $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ is trivial, as desired. □

The following are direct consequences of the above lemmas, whose proof is left to the readers.

Corollary 5.12 ($\mathrm{Gr}_R(\widetilde{\mathcal{M}})$ as Zariski fibration)**.** $\mathrm{Gr}^n_R(\widetilde{\mathcal{M}})$ *is a Zariski fibration of over* S *with fibers isomorphic to* $\mathrm{Gr}^n_R(\widetilde{R}^d)$.

Corollary 5.13 ($\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}})$ as constructible fibration)**.** *Consider a pair of* $\widetilde{R}$*-lattices* $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$. *Then there is a stratification* $\mathrm{Spec}\, S = \bigsqcup_{\alpha\in\mathbf{A}} X_\alpha$ *trivializing* $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$*, i.e., for each* α*, there exists a pair of lattices* $\underset{\sim}{M}_\alpha \subseteq \widetilde{M}_\alpha$ *over* k *whose base change to* X_α *is isomorphic to* $\underset{\sim}{\mathcal{M}}_{X_\alpha} \subseteq \widetilde{\mathcal{M}}_{X_\alpha}$.

5.2. **Loop space interpretations.** For Y/k an affine variety, define functors

$$L^K Y : \mathbf{Alg}_k \to \mathbf{Set},\ S \mapsto Y(S\widehat{\otimes}K), \tag{5.6}$$

$$L^R Y : \mathbf{Alg}_k \to \mathbf{Set},\ S \mapsto Y(S\widehat{\otimes}R). \tag{5.7}$$

In this section, we will exclusively work in the setting $R = \widetilde{R}$. For example, when $\widetilde{R} = k[[T]]$, these definitions recover the classical definition of loop and arc spaces LY and L^+Y, cf. [64]. When $\widetilde{R}$ is a product of s copies of $k[[T]]$'s, then $L^K X$ *resp.* $L^{\widetilde{R}} X$ is just a product of s copies of LY *resp.* L^+Y.

5.2.1. *Main players.* We will begin by assuming that $s = 1$, i.e., $\widetilde{R} = k[[T]]$. Let V be a d-dimensional k-vector space and let $\mathrm{GL}(V)$ be the general linear group over V. When the context is clear, we will just write the group as GL. It is classically known that $L^K\,\mathrm{GL}$ is an ind-scheme, while $L^{\widetilde{R}}\,\mathrm{GL}$ is a pro-algebraic group. We refer the readers to [57, Appendix A] for a general discussion of pro-algebraic groups, their actions on ind-schemes, and torsors under pro-algebraic groups.

For our purpose, we also introduce several ind-subschemes of $L^K\,\mathrm{GL}$. Let $L^K_+\,\mathrm{GL}$ be the closed ind-subscheme whose S-points are elements in $\mathrm{GL}(S\widehat{\otimes}K)$ with entries lying in $S\widehat{\otimes}\widetilde{R}$. Note that $L^K_+\,\mathrm{GL}$ is only a semigroup.

For a flag $\mu : W \subseteq V$, there is a parabolic subgroup $P_\mu \subseteq \mathrm{GL}$ fixing μ. Let $\mathcal{K}_\mu$ be the kernel of the natural morphism $P_\mu \to \mathrm{GL}(V/W)$. The construction yields an ind-closed subgroup $L^K\mathcal{K}_\mu \subseteq L^K\,\mathrm{GL}$ and a subsemigroup $L^K_+\mathcal{K}_\mu := L^K\mathcal{K}_\mu \cap L^K_+\,\mathrm{GL}$. It also gives rise to a closed pro-subgroup $L^{\widetilde{R}}\mathcal{K}_\mu \subseteq L^{\widetilde{R}}\,\mathrm{GL}$. Note that $L^{\widetilde{R}}\mathcal{K}_\mu = L^K_+\mathcal{K}_\mu \cap L^{\widetilde{R}}\,\mathrm{GL}$.

Fix an identification $V_K \simeq K^d$. Then $L^K\,\mathrm{GL}$ acts on K^d. From this, there is a left action $L^K\,\mathrm{GL} \curvearrowright \mathrm{Gr}_{\widetilde{R}}$ given by $g \cdot \mathcal{L} \to g\mathcal{L}$ over S-points. Fixing an $x_0 \in \mathrm{Gr}_{\widetilde{R}}(k)$, the action $L^K\,\mathrm{GL} \curvearrowright \mathrm{Gr}_{\widetilde{R}}$ induces an isomorphism $L^K\,\mathrm{GL}/L^{\widetilde{R}}\,\mathrm{GL} \simeq \mathrm{Gr}_{\widetilde{R}}$ sending the identity element $e \in L^K\,\mathrm{GL}(k)$ to x_0. It is also classically known that $L^K\,\mathrm{GL}$ is a Zariski $L^{\widetilde{R}}\,\mathrm{GL}$-torsor over $\mathrm{Gr}_{\widetilde{R}}$, since the universal lattice over $\mathrm{Gr}_{\widetilde{R}}$ is Zariski locally free.

The constructions in the $s \geq 1$ case are nothing other than taking products of the constructions in the $s = 1$ case. Let $V_1, \ldots, V_s$ be d-dimensional vector spaces. We can identify $L^K\,\mathrm{GL}$ *resp.* $L^{\widetilde{R}}\,\mathrm{GL}$ as $\prod L^{K_i}\,\mathrm{GL}(V_i)$ *resp.* $\prod L^{\widetilde{R}_i}\,\mathrm{GL}(V_i)$. Then again, we have the notion of $L^K_+\,\mathrm{GL}$, which is simply the product of several $L^{K_i}_+\,\mathrm{GL}$. Furthermore, for each i, let $\mu_i : W_i \subseteq V_i$ be a flag, and let $\mu = (\mu_1, \ldots, \mu_s)$.

Then we define $L^K\mathcal{K}_\mu$ *resp.* $L^K_+\mathcal{K}_\mu$ *resp.* $L^{\widetilde{R}}\mathcal{K}_\mu$ to be the product of several $L^{K_i}\mathcal{K}_{\mu_i}$ *resp.* $L^{K_i}_+\mathcal{K}_{\mu_i}$ *resp.* $L^{\widetilde{R}_i}\mathcal{K}_{\mu_i}$. If we fix an identification $V_{i,K_i} \simeq K_i^d$ for each i, then there is a left action $L^K\,\mathrm{GL} \curvearrowright \mathrm{Gr}_{\widetilde{R}}$ given by $g\cdot\mathcal{L} \to g\mathcal{L}$ over S-points. Fixing an $x_0 \in \mathrm{Gr}_{\widetilde{R}}(k)$, the action $L^K\,\mathrm{GL} \curvearrowright \mathrm{Gr}_{\widetilde{R}}$ induces an isomorphism $L^K\,\mathrm{GL}/L^{\widetilde{R}}\,\mathrm{GL} \simeq \mathrm{Gr}_{\widetilde{R}}$ sending the identity element $e \in L^K\,\mathrm{GL}(k)$ to x_0.

5.2.2. *Loop space interpretations on padding cells.* The following lemma is straightforward, and is left to the readers:

Lemma 5.14. *Notation as above. Suppose that $\underset{\sim}{M} \subseteq \widetilde{M}$ is a pair of $\widetilde{R}$-lattices. Let $\underset{\sim}{M}_i \subseteq \widetilde{M}_i$ be its component over $\widetilde{R}_i$. For each i, let $r_i = \mathrm{rk}_i(\widetilde{M}/\underset{\sim}{M})$ be as in Lemma 4.4. Then there exists an $\widetilde{R}_i$-basis $\{v_{i,1}, v_{i,2}, \ldots, v_{i,d}\}$ of $\widetilde{M}_i$, such that*

- (a) *$\{\overline{v}_{i,1}, \overline{v}_{i,2}, \ldots, \overline{v}_{i,r_i}\}$ is a minimal set of generators of $\widetilde{M}_i/\underset{\sim}{M}_i$,*
- (b) *$\{v_{i,r_i+1}, \ldots, v_{i,d}\}$ is contained in $\underset{\sim}{M}_i$.*

If we identify V_i with $\mathrm{span}_k\{v_{i,1}, v_{i,2}, \ldots, v_{i,d}\}$, and let μ_i be the flag $W_i := \mathrm{span}_k\{v_{i,r_i+1}, \ldots, v_{i,d}\} \subseteq V_i$. Then we have left actions $L^K_+\mathcal{K}_\mu \curvearrowright \mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$ and $L^K_+\mathcal{K}_\mu \curvearrowright \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{M}, \widetilde{M})$ by simply acting on the basis.

Proposition 5.15. *Notation as above. Let $\underset{\sim}{M} \subseteq \widetilde{M}$ be a pair of $\widetilde{R}$-lattices. Pick bases and make identifications as in Lemma 5.14. We have the following commuting diagram:*

$$\begin{array}{ccc}
L^K_+\mathcal{K}_\mu/L^{\widetilde{R}}\mathcal{K}_\mu & \xrightarrow{\sim} & \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{M}, \widetilde{M}) \\
\big\downarrow & & \big\downarrow \\
L^K_+\,\mathrm{GL}/L^{\widetilde{R}}\,\mathrm{GL} & \xrightarrow{\sim} & \mathrm{Gr}_{\widetilde{R}}(\widetilde{M}) \\
\big\downarrow & & \big\downarrow \\
L^K\,\mathrm{GL}/L^{\widetilde{R}}\,\mathrm{GL} & \xrightarrow{\sim} & \mathrm{Gr}_{\widetilde{R}}
\end{array} \tag{5.8}$$

The isomorphisms send the identity e to $\widetilde{M}$. Furthermore, $L^K_+\,\mathrm{GL}$ is a Zariski $L^{\widetilde{R}}\,\mathrm{GL}$-torsor over $\mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$, and $L^K_+\mathcal{K}_\mu$ is a Zariski $L^{\widetilde{R}}\mathcal{K}_\mu$-torsor over $\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{M}, \widetilde{M})$.

Proof. Without loss of generality, we can assume that $s = 1$. The isomorphism on the third row of (5.8) is classically known. For the isomorphism on the second row, it is enough to check that $L^K_+\,\mathrm{GL}/L^{\widetilde{R}}\,\mathrm{GL}$ and $\mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$ are the same sheaf. This follows since if $\widetilde{\mathcal{L}}$ is free $\widetilde{R}$-lattice over an affine scheme S, then there exists an element $g \in L^K_+\,\mathrm{GL}(S)$ such that $g\widetilde{M}_S = \widetilde{\mathcal{L}}$, and any such two elements differ by an element in $L^{\widetilde{R}}\,\mathrm{GL}(S)$. Since the universal lattice over $\mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$ is Zariski locally free, we get the Zariski torsor assertion.

For the isomorphism on the first row of (5.8), consider the action $L^K_+\,\mathrm{GL} \curvearrowright \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{M}, \widetilde{M})$ as in Lemma 5.14. Let $\widetilde{\mathcal{L}}$ be a free $\widetilde{R}$-lattice over an affine scheme S, with the property that $\widetilde{\mathcal{L}} + \underset{\sim}{M}_S = \widetilde{M}_S$. Start with an $S\widehat{\otimes}\widetilde{R}$-basis $\{w_1, w_2, \ldots, w_d\}$ of $\widetilde{\mathcal{L}}$, and write $w_j = \sum_{i=1}^d a_{ij}v_i$ with $a_{ij} \in S\widehat{\otimes}\widetilde{R} = S[[T]]$. Since $w_1, \ldots, w_d$ generate $\widetilde{M}_S/\underset{\sim}{M}_S$, and $\widetilde{M}_S/\underset{\sim}{M}_S \otimes S[[T]]/(T)$ is free of rank r over S, we may assume after passing to an affine cover of S that $(a_{ij})_{1\le i,j\le r}$ is invertible mod T, thus invertible itself. By column operation, we may assume $(a_{ij})_{1\le i\le r, 1\le j\le d} = [\mathrm{Id}_r\ O_{r\times d-r}]$. This shows the existence of $g \in L^K_+\mathcal{K}_\mu(S)$ such that $g\widetilde{M}_S = \widetilde{\mathcal{L}}$. Any such two elements differ by an element in $L^K_+\mathcal{K}_\mu(S) \cap L^{\widetilde{R}}\,\mathrm{GL}(S) = L^{\widetilde{R}}\mathcal{K}_\mu(S)$. The Zariski torsor assertion is similar to the second row. □

Corollary 5.16 (The trivial relative case)**.** *Let S be a k-algebra. Let $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ be a trivial pair of $\widetilde{R}$-lattices, i.e., it is the base change of an inclusion $\underset{\sim}{M} \subseteq \widetilde{M}$ of lattices over k along the structural* *morphism $k \to S$. Picking bases as in Lemma 5.14, we have the following commuting diagram:*

$$\begin{array}{ccccc} (L^K_+\mathcal{K}_\mu/L^{\widetilde{R}}\mathcal{K}_\mu)_S & \xrightarrow{\sim} & \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{M},\widetilde{M})_S & \xrightarrow{\sim} & \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}},\widetilde{\mathcal{M}}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (L^K_+\,\mathrm{GL}\,/L^{\widetilde{R}}\,\mathrm{GL})_S & \xrightarrow{\sim} & \mathrm{Gr}_{\widetilde{R}}(\widetilde{M})_S & \xrightarrow{\sim} & \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (L^K\,\mathrm{GL}\,/L^{\widetilde{R}}\,\mathrm{GL})_S & \xrightarrow{\sim} & \mathrm{Gr}_{\widetilde{R},S} & = & \mathrm{Gr}_{\widetilde{R},S} \end{array} \tag{5.9}$$

The isomorphisms send the identity section $\underline{e}$ of $(L^K_+\mathcal{K}_\mu/L^{\widetilde{R}}\mathcal{K}_\mu)_S$ to $[\widetilde{M}]_S$ and then to $\widetilde{\mathcal{M}}$. Furthermore, (5.9) is functorial in S.

Proof. This follows from Proposition 5.15 and the trivial cases of Corollaries 5.12 and 5.13. ☐

Remark 5.17. At the beginning of §5.1.2, we have already used the notation $\underline{e}$ for the distinguished S-point of $\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}},\widetilde{\mathcal{M}})$ corresponding to $\widetilde{\mathcal{M}}$. Corollary 5.16 justifies the usage.

5.3. **Constructible geometry.** We set up the language for studying geometry up to stratifications. Recall that for a scheme X, a Zariski stratification (or simply, stratification) is a morphism $\breve{X} = \bigsqcup_{\alpha\in\mathbf{A}} X_\alpha \to X$ that is bijective on field valued points such that $\mathbf{A}$ is a finite set, and each $X_\alpha \to X$ is a locally closed subscheme. A stratification is a categorical monomorphism in **Sch**, a composition of stratifications is a stratification, and moreover, the pullback of a stratification $\breve{X} \to X$ along a map $Y \to X$ is a stratification of Y. These facts show that the set of stratifications satisfy the conditions RMS1∼RMS3 of [60, 04VC]. Therefore, we can localize at the set of stratifications, and obtain a category where stratifications are isomorphisms, cf. [60, 04VH]. The resulting category is called the **category of constructible schemes**, denoted **CSch**. If Z is a scheme, the slice category $\mathbf{CSch}_Z := (\mathbf{CSch} \downarrow Z)$ is called the **category of constructible Z-schemes**.

A morphism in $\mathrm{Hom}_{\mathbf{CSch}}(X,Y)$ will be called a **constructible morphism** and written as $f : X \xrightarrow{c} Y$. It is represented by a roof $X \leftarrow \breve{X} \xrightarrow{\breve{f}} Y$, where $\breve{X} \to X$ is a stratification and $\breve{f}$ is a morphism of schemes. When the context is clear, we will just write the roof as $\breve{f} : \breve{X} \to Y$. If we represent $f : X \xrightarrow{c} Y$ *resp.* $g : Y \xrightarrow{c} Z$ as $\breve{f} : \breve{X} \to Y$ *resp.* $\breve{g} : \breve{Y} \to Z$, then the composition of $g \circ f$ can be represented as $\breve{g} \circ p_{\breve{Y}} : \breve{X} \times_Y \breve{Y} \to Z$, where $p_{\breve{Y}}$ is the projection $\breve{X} \times_Y \breve{Y} \to \breve{Y}$, which is the base change of $\breve{f}$ along $\breve{Y} \to Y$. Two representations $\breve{X} \to Y$ and $\breve{X}' \to Y$ are called equivalent, if they represent the same constructible morphism. More concretely, they are equivalent if they become the same after passing to a common refinement.

Remark 5.18 (Field valued points)**.** Let F be a field. There is a functor **Sch** → **Set** by taking F-points. Since stratifications induces bijections on F-points, we can localize and obtain a functor **CSch** → **Set**. This is called "taking field valued points of constructible schemes". As a variant, we have a functor **CSch** → **Set** sending a constructible scheme to its underlying set.

Remark 5.19 (Constructible subschemes)**.** A constructible morphism $f : X \xrightarrow{c} Y$ in **CSch** is called a **constructible subscheme** *resp.* **constructible immersion**, if we can represent it as a morphism $\breve{f} : \bigsqcup_{\alpha\in\mathbf{A}} X_\alpha \to Y$, such that each $\breve{f}|_{X_\alpha} : X_\alpha \to Y$ is a locally closed subscheme *resp.* immersion. When f is a constructible subscheme, we will also write $X \overset{c}{\subseteq} Y$.

If $X_i \xrightarrow{c} Y, i = 1, 2$ are constructible subschemes such that $X_1 \overset{c}{\subseteq} X_2$ and $X_2 \overset{c}{\subseteq} X_1$, we say that X_1 is **constructibly identical** to X_2 (as constructible subschemes of Y). In this case, we will write $X_1 \overset{c}{=} X_2$. For example, any stratification $\breve{X} \to X$ is constructibly identical to X. Note that a constructible identity is necessarily a constructible isomorphism and is an equivalence relation. If F is a field, we will naturally identify the F-points of an $X \overset{c}{\subseteq} Y$ as a subset of the F-points of Y. Then constructibly identical constructible subschemes of Y have the same F-points. Furthermore, their underlying sets are identical (as subsets of the underlying set $|Y|$ of Y).

Conversely, it is easy to see that a constructible subscheme of Y is determined by its underlying set up to constructible identity. Moreover, a subset of $|Y|$ which is a finite disjoint union of locally closed subspaces of $|Y|$ (i.e., a constructible subset, at least when Y is Noetherian) can be upgraded to a constructible subscheme. Say, if S is a locally closed topological subspace which is the intersection of a closed set F and an open set O, then we give O the unique open subscheme structure of Y and give S the unique structure as a reduced closed subscheme of O. Therefore, we have the following equivalence:

Lemma 5.20.

$$\left\{\begin{matrix}\textit{Constructible subschemes of } Y\\ \textit{up to constructible identity}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Subsets of } |Y| \textit{ which are finite disjoint}\\ \textit{unions of locally closed subspaces}\end{matrix}\right\}, \tag{5.10}$$
$$X \to |X|.$$

In particular, two constructible subschemes of Y are constructibly identical if they have the same underlying set (or if they have the same field valued points).

Remark 5.21 (Fiber products and fibrations)**.** Let $f : X \overset{c}{\to} W$ and $g : Y \overset{c}{\to} W$ be in $\mathbf{CSch}_Z$. We can represent f, g by $\breve{f} : \breve{X} \to W$ and $\breve{g} : \breve{Y} \to W$. Then $\breve{X} \times_W \breve{Y}$ satisfies the desired universal property of fiber product. A constructible morphism $f : X \overset{c}{\to} W$ in $\mathbf{CSch}_Z$ is said to be a **constructible fibration**, if there is a stratification $\breve{W} = \bigsqcup_{\beta\in\mathbf{B}} W_\beta \to W$, such that for each $\beta \in \mathbf{B}$, $X \times_W W_\beta \overset{c}{\simeq} Y_\beta \times_Z W_\beta$ for some $Y_\beta \in \mathbf{CSch}_Z$. If $Y_\beta \overset{c}{\simeq} Y$ for all $\beta \in \mathbf{B}$, then $X \overset{c}{\simeq} Y \times_Z W$. For example, if in $\mathbf{Sch}_k$, X is a Zariski GL_n-torsor over a quasi-compact base scheme W, then in $\mathbf{CSch}_k$ we simply have $X \overset{c}{\simeq} W \times_k \mathrm{GL}_n$.

Remark 5.22 (Graphs)**.** Let $f : X \overset{c}{\to} Y$ be in $\mathbf{CSch}_Z$. The relative graph $\Gamma_Z(f)$ is the constructible morphism $\mathrm{id}_X \times_Z f : X \overset{c}{\to} X \times_Z Y$. Note that projection induces a constructible isomorphism $\Gamma_Z(f) \overset{c}{\simeq} X$. It is a constructible subscheme. When the context is clear, we will often omit the subscript Z from $\Gamma_Z(f)$.

Remark 5.23 (Classes in Grothendieck ring)**.** If $X \overset{c}{\simeq} Y$ in $\mathbf{CSch}_Z$, then X, Y have the same class in $K_0(\mathbf{Sch}_Z)$.

5.3.1. *Constructible ind-geometry.* We can generalize constructible geometry to the setting of ind-schemes. The following definitions may not be the most general one, but they serve our purpose well:

Definition 5.24. Let X, Y and Z be ind-schemes,

(a) Write $X = \bigcup_i X_i$ as a direct union of schemes, such that each $X_i \hookrightarrow X_{i+1}$ is a closed immersion. We say that $\breve{X} = \bigsqcup_{\alpha\in\mathbf{A}} X_\alpha \to X$ is a stratification ($\mathbf{A}$ may be an infinite set), if (1) it is a decomposition of the underlying topological spaces, (2) each X_α is a locally closed subscheme of X and (3) for each i, $\breve{X}_i = \bigsqcup_\alpha X_\alpha \cap X_i \to X_i$ is a stratification of scheme (so there are only finitely many $\alpha \in \mathbf{A}$ such that $|X_\alpha \cap X_i| \neq \emptyset$). We say that $\breve{X}$ is an affine stratification, if each X_α is affine.
(b) Once we have the notion of stratification, we can again do localization and let **CIndSch** and $\mathbf{CIndSch}_Z$ be the category of constructible ind-schemes and constructible ind-Z-schemes. By taking the direct union, one can easily generalize the notions in Remark 5.18∼5.21 to the setting of constructible ind-schemes. More precisely, we have the notion of constructible ind-subschemes and their fiber products and graphs.

We will need the following generalization of Lemma 5.20:

Lemma 5.25. *Two constructible ind-subschemes of a constructible ind-subscheme Y are constructibly identical if they have the same underlying set (or the same field valued points).*

Proof. Pass Lemma 5.20 to the direct limit. □

5.3.2. *Constructible lattices.*

Definition 5.26. A constructible R-lattice over an ind-scheme X is a coherent $\mathcal{O}_{\check{X}}\widehat{\otimes}R$-module $\mathcal{F}$ over an affine stratification $\check{X} = \bigsqcup_{\alpha\in\mathbf{A}} X_\alpha \to X$, which is a lattice over each X_α.

Lemma 5.27. *Let X be a quasi-compact k-scheme. Then any coherent $\mathcal{O}_X\widehat{\otimes}R$-submodule $\mathcal{F} \subseteq \mathcal{O}_X\widehat{\otimes}K^d$ such that $\mathcal{F}\otimes(\mathcal{O}_X\widehat{\otimes}K) = \mathcal{O}_X\widehat{\otimes}K^d$ is a constructible lattice. In particular, it induces a constructible morphism $f : X \overset{c}{\to} \mathrm{Gr}_R$ such that $f^*\mathcal{U} = \mathcal{F}$. This lemma extends to the situation where X is an ind-scheme with X_i quasi-compact over k.*

Proof. Using quasi-compactness, we can reduce to the case where X is affine. Since $\mathcal{F}$ is coherent, there is an i such that $\underline{T}^i\mathcal{O}_X\widehat{\otimes}\widetilde{R} \subseteq \mathcal{F} \subseteq \underline{T}^{-i}\mathcal{O}_X\widehat{\otimes}\widetilde{R}$. The quotient $(\underline{T}^{-i}\mathcal{O}_X\widehat{\otimes}\widetilde{R})/\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. Put a flattening stratification on X, i.e., $X = \bigsqcup_\alpha X_\alpha$ such that $(\underline{T}^{-i}\mathcal{O}_X\widehat{\otimes}\widetilde{R})/\mathcal{F}$ is flat over each X_α. The stratification is finite since the rank of $(\underline{T}^{-i}\mathcal{O}_X\widehat{\otimes}\widetilde{R})/\mathcal{F}$ at closed points are bounded above by the rank of $(\underline{T}^{-i}\mathcal{O}_X\widehat{\otimes}\widetilde{R})/(\underline{T}^{i}\mathcal{O}_X\widehat{\otimes}\widetilde{R})$. Then $\mathcal{F}$ is a rank d lattice over each X_α, and we get a morphism $f_\alpha : X_\alpha \to \mathrm{Gr}_R$ such that the pullback of $f_\alpha^*\mathcal{U}$ is $\mathcal{F}_{X_\alpha}$. □

5.4. **Lattice closure and summation as constructible fibrations.**

5.4.1. *Lattice closure.* Let $\pi : \operatorname{Spec}\widetilde{R} \to \operatorname{Spec} R$. Consider the universal lattice $\mathcal{U}$ over Gr_R. Passing Lemma 5.27 to the inductive limit, the coherent sheaf $\widetilde{R}\mathcal{U} = (1\widehat{\otimes}\pi)^*\mathcal{U}$ is a constructible $\widetilde{R}$-lattices over Gr_R, and it induces a constructible morphism

$$\underline{\pi}^* : \mathrm{Gr}_R \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}. \tag{5.11}$$

Over k-points, this coincides with the map sending an R-lattice L to $\widetilde{R}L$, as in §4.2.

The graph $\Gamma(\underline{\pi}^*) \overset{c}{\subseteq} \mathrm{Gr}_R \times \mathrm{Gr}_{\widetilde{R}}$ can be thought of as the extension fiber of the universal lattice over $\mathrm{Gr}_{\widetilde{R}}$. In particular, for any $\widetilde{R}$-lattice $\widetilde{L}$ over a field, the extension fiber $E_R(\widetilde{L})$ from §4.2 admits a constructible scheme structure given by $\Gamma(\underline{\pi}^*) \times_{\mathrm{Gr}_{\widetilde{R}}} x$ (or $(\underline{\pi}^*)^{-1}(x)$), where $x \to \mathrm{Gr}_{\widetilde{R}}$ is the point corresponding to $\widetilde{L}$. Let $E_R^n(\widetilde{L})$ be the intersection of $E_R(\widetilde{L})$ with $\mathrm{Gr}_R^n(\widetilde{L})$.

If X is a k-scheme, we also have a trivial constructible X-morphism $\underline{\pi}_X^* := \underline{\pi}^* \times \mathrm{id}_X : \mathrm{Gr}_{R,X} \overset{c}{\to} \mathrm{Gr}_{\widetilde{R},X}$. We write $\Gamma(\underline{\pi}_X^*) \overset{c}{\subseteq} \mathrm{Gr}_R \times \mathrm{Gr}_{\widetilde{R}} \times X$ for its relative graph.

Theorem 5.28 (Lattice closure as a constructible fibration)**.** *Fix an rank d $\widetilde{R}$-lattice $\widetilde{M}$. Then $\underline{\pi}^* : \mathrm{Gr}_R \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}$ is a constructible fibration with constant fiber $E_R(\widetilde{M})$. As a result, $\mathrm{Gr}_R \overset{c}{\simeq} \mathrm{Gr}_{\widetilde{R}} \times E_R(\widetilde{M})$. Since $\widetilde{M} \simeq \widetilde{R}^d$, we also have $\mathrm{Gr}_R \overset{c}{\simeq} \mathrm{Gr}_{\widetilde{R}} \times E_R(\widetilde{R}^d)$.*

Proof. The strategy is to use loop group action to permute the extension fibers. Since $\mathrm{Gr}_R \overset{c}{\simeq} \Gamma(\underline{\pi}^*)$, we will show that the projection map $p_2 : \Gamma(\underline{\pi}^*) \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}$ is a constructible fibration with constant fiber $E_R(\widetilde{M})$. Let $x_0 \in \mathrm{Gr}_{\widetilde{R}}(k)$ be the point corresponding to $\widetilde{M}$. Identify $L^K \mathrm{GL}/L^{\widetilde{R}}\mathrm{GL} \simeq \mathrm{Gr}_{\widetilde{R}}$, which sends the identity $e \in L^K\mathrm{GL}(k)$ to x_0. Because $L^K \mathrm{GL}$ is a Zariski $L^{\widetilde{R}}\mathrm{GL}$-torsor over $\mathrm{Gr}_{\widetilde{R}}$, $\mathrm{Gr}_{\widetilde{R}}$ is covered by open affine subschemes W which admit sections $\sigma : W \to (L^K\mathrm{GL})_W := L^K\mathrm{GL} \times_{\mathrm{Gr}_{\widetilde{R}}} W$. Let $\underline{\pi}_W^*$ be the restriction of $\underline{\pi}^*$ to $(\underline{\pi}^*)^{-1}W$. In the following, we will show that $\Gamma(\underline{\pi}_W^*) \overset{c}{\simeq} E_R(\widetilde{M}) \times W$. Since $\Gamma(\underline{\pi}_W^*) \overset{c}{\simeq} \Gamma(\underline{\pi}^*) \times_{\mathrm{Gr}_{\widetilde{R}}} W$, this will immediately imply that p_2 is a constructible fibration.

Recall that $L^K\mathrm{GL}$ acts on Gr_R. Consider the following isomorphism of ind-schemes:

$$\widetilde{\sigma} : \mathrm{Gr}_R \times W \to \mathrm{Gr}_R \times W,\ (\mathcal{L}, w) \to (\sigma(w)\mathcal{L}, w). \tag{5.12}$$

Let $U \subseteq \mathrm{Gr}_R \times W$ be the image under $\widetilde{\sigma}$ of the constructible subscheme $E_R(\widetilde{M}) \times W$, then U is again a constructible subscheme of $\mathrm{Gr}_R \times W$. Let F be a field, then the F-points of U are pairs $(L, \widetilde{L})$ such that $\widetilde{L}$ is an F-point of W and L is an R-lattice over F with closure $\widetilde{L}$. Therefore $U \overset{c}{=} \Gamma(\underline{\pi}_W^*)$ as constructible subschemes of $\mathrm{Gr}_R \times W$, since they have the same closed points. As a result, $E_R(\widetilde{M}) \times W \overset{c}{\simeq} \Gamma(\underline{\pi}_W^*)$. □

Corollary 5.29. *let X be a quasi-compact k-schemes and $\widetilde{\mathcal{M}}$ be a rank d X-family of $\widetilde{R}$-lattice. Then $\mathrm{Gr}_R(\widetilde{\mathcal{M}}) \overset{c}{\simeq} X \times \mathrm{Gr}_{\widetilde{R}}(\widetilde{R}^d) \times E_R(\widetilde{R}^d)$. The isomorphism also admits a graded version: $\mathrm{Gr}_R^n(\widetilde{\mathcal{M}}) \overset{c}{\simeq} X \times \bigsqcup_{i+j=n} \mathrm{Gr}^i_{\widetilde{R}}(\widetilde{R}^d) \times E^j_R(\widetilde{R}^d)$.*

Proof. By Corollary 5.12, it suffices to treat the case where $X = \operatorname{Spec} k$. In the following, we will write $\widetilde{M}$ instead of $\widetilde{\mathcal{M}}$. Note that $\mathrm{Gr}_R(\widetilde{M}) \overset{c}{=} (\underline{\pi}^*)^{-1}\,\mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$. This can be checked over field valued points. Then we can base changing the fibration in Theorem 5.28 along the closed embedding $\mathrm{Gr}_{\widetilde{R}}(\widetilde{M}) \subseteq \mathrm{Gr}_{\widetilde{R}}$ to get a contructible fibration structure of $\underline{\pi}^*|_{\mathrm{Gr}_R(\widetilde{M})} : \mathrm{Gr}_R(\widetilde{M}) \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}(\widetilde{M})$, which says that $\mathrm{Gr}_R(\widetilde{M}) \overset{c}{\simeq} \mathrm{Gr}_{\widetilde{R}}(\widetilde{M}) \times E_R(\widetilde{M})$. The graded version is easy, and is left to the readers. □

5.4.2. *Lattice summation.* Suppose that X is a quasi-compact k-scheme. If $\mathcal{M}'$ is an R-lattice over X, then the sheaf $p_1^*\mathcal{U} + p_2^*\mathcal{M}'$ over $\mathrm{Gr}_{R,X}$ is a constructible lattice, where as usual, p_1 *resp.* p_2 denotes the projection of $\mathrm{Gr}_{R,X}$ to Gr_R *resp.* X. Therefore, it induces by Lemma 5.27 (and taking inductive limit) a constructible map of X-ind-schemes $\mathrm{Sum}_{\mathcal{M}'} : \mathrm{Gr}_{R,X} \overset{c}{\to} \mathrm{Gr}_R([\mathcal{M}', \infty))$. The map sends a lattice $\mathcal{L}$ to its summation with $\mathcal{M}'$, which is, up to stratification, another lattice. We write $\Gamma(\mathrm{Sum}_{\mathcal{M}'}) \overset{c}{\subseteq} \mathrm{Gr}_{R,X} \times_X \mathrm{Gr}_R([\mathcal{M}', \infty)) \subseteq \mathrm{Gr}_R \times \mathrm{Gr}_R \times X$ for the relative graph of $\mathrm{Sum}_{\mathcal{M}'}$.

Lemma 5.30. *Consider the pair of lattices $p_2^*\mathcal{M}' \subseteq p_1^*\mathcal{U}$ over $\mathrm{Gr}_R([\mathcal{M}', \infty))$. Then $\Gamma(\mathrm{Sum}_{\mathcal{M}'}) \overset{c}{=} \mathrm{Pd}_R(p_2^*\mathcal{M}', p_1^*\mathcal{U})$ as constructible ind-subschemes of $\mathrm{Gr}_{R,X} \times_X \mathrm{Gr}_R([\mathcal{M}', \infty))$. In particular, the projection to the first coordinate induces a constructible isomorphism $\mathrm{Pd}_R(p_2^*\mathcal{M}', p_1^*\mathcal{U}) \overset{c}{\simeq} \mathrm{Gr}_{R,X}$.*

Proof. By Lemma 5.25, it suffices to check that the underlying sets of $\Gamma(\mathrm{Sum}_{\mathcal{M}'})$ and $\mathrm{Pd}_R(p_2^*\mathcal{M}', p_1^*\mathcal{U})$ coincide. Let F denote a field extension of k. It suffices to check that they have the same F-points. By base change to a closed point of X, it suffices to assume $X = \operatorname{Spec} F$. Then $\mathcal{M}'$ and $\mathcal{M}$ are simply R-lattices over F. The F-points of $\Gamma(\mathrm{Sum}_{\mathcal{M}'})$ are pairs $\{(L, L + \mathcal{M}') | L \in \mathrm{Gr}_R(F)\}$, while the F-points of $\mathrm{Pd}_R(p_2^*\mathcal{M}', p_1^*\mathcal{U})$ are $\{(L, L') | L \in \mathrm{Gr}_R(F), L' \in \mathrm{Gr}_R([\mathcal{M}', \infty))(F), L + \mathcal{M}' = L'\}$. Clearly, they are the same set. □

Theorem 5.31 (Lattice summation as a constructible fibration)**.** *Let X be a quasi-compact k-scheme and $\underset{\sim}{\mathcal{M}}$ be an X-family of $\widetilde{R}$-lattices. Then $\mathrm{Sum}_{\underset{\sim}{\mathcal{M}}} : \mathrm{Gr}_{\widetilde{R},X} \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty))$ is a constructible fibration.*

Proof. By Lemma 5.30, we have $\mathrm{Gr}_{\widetilde{R},X} \overset{c}{\simeq} \Gamma(\mathrm{Sum}_{\mathcal{M}'}) \overset{c}{=} \mathrm{Pd}_{\widetilde{R}}(p_2^*\underset{\sim}{\mathcal{M}}, p_1^*\widetilde{\mathcal{U}})$. So it suffices to show that the projection $\pi_2 : \mathrm{Pd}_{\widetilde{R}}(p_2^*\underset{\sim}{\mathcal{M}}, p_1^*\widetilde{\mathcal{U}}) \overset{c}{\to} \mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty))$ is a constructible fibration. By Corollary 5.13, there is a stratification $\mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty)) \overset{c}{=} \bigsqcup_{\beta \in \mathbf{B}} Y_\beta$, such that for each β, the restriction of $p_2^*\underset{\sim}{\mathcal{M}} \subseteq p_1^*\widetilde{\mathcal{U}}$ to Y_β is trivial. Say, we have an inclusion $\underset{\sim}{N}_\beta \subseteq \widetilde{N}_\beta$ of lattices over k with the property that $(\underset{\sim}{N}_\beta \subseteq \widetilde{N}_\beta)_{Y_\beta} \simeq (p_2^*\underset{\sim}{\mathcal{M}} \subseteq p_1^*\widetilde{\mathcal{U}})_{Y_\beta}$. Then we have a constructible fibration

$$\begin{array}{ccccc}
\mathrm{Gr}_{\widetilde{R},X} & \xrightarrow{\overset{c}{\sim}} & \bigsqcup_{\beta \in \mathbf{B}} \mathrm{Pd}_{\widetilde{R}}(p_2^*\underset{\sim}{\mathcal{M}}_{Y_\beta}, p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) & \xrightarrow{\overset{c}{\sim}} & \bigsqcup_{\beta \in \mathbf{B}} \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta) \times Y_\beta \\
{\scriptstyle \mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}}\downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty)) & \xrightarrow{\overset{c}{\sim}} & \bigsqcup_{\beta \in \mathbf{B}} Y_\beta & = & \bigsqcup_{\beta \in \mathbf{B}} Y_\beta
\end{array} \tag{5.13}$$

□

Remark 5.32. Recall that $\mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty))$ is a closed ind-subscheme of $\mathrm{Gr}_{\widetilde{R},X}$. Let ι be this embedding, then $\mathrm{Sum}_{\mathcal{M}'} \circ \iota = \mathrm{id}$. In (5.13), after passing to stratification, $\iota|_{Y_\beta}$ corresponds to the embedding of Y_β to $\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta) \times Y_\beta$ via the distinguished k-point $\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta)$ corresponding to $\widetilde{N}_\beta$.

Remark 5.33 (Loop space interpretation)**.** Corollary 5.16 tells that for each β, after choosing a certain basis of $\widetilde{N}_\beta$ and picking a certain flag μ_β as in Lemma 5.14, there are isomorphisms

$$(L_+^K \mathcal{K}_{\mu_\beta} / L^{\widetilde{R}} \mathcal{K}_{\mu_\beta}) \times Y_\beta \simeq \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta) \times Y_\beta \simeq \mathrm{Pd}_{\widetilde{R}}(p_2^* \underset{\sim}{\mathcal{M}}_{Y_\beta}, p_1^* \widetilde{\mathcal{U}}_{Y_\beta}) \tag{5.14}$$

sending the identity section $\underline{e}_\beta$ to $[\widetilde{N}_\beta] \times Y_\beta$ then to $p_1^* \widetilde{\mathcal{U}}_{Y_\beta}$.

5.5. **Restricted lattice closure as a relative constructible fibration.** Suppose that X is a quasi-compact k-scheme and $\mathcal{M}$ is an X-family of R-lattices. Up to stratification of X, we can assume that the closure $\widetilde{R}\mathcal{M}$ is already an $\widetilde{R}$-lattice (Lemma 5.27). Let $\widetilde{\mathcal{M}} := \widetilde{R}\mathcal{M}$ and let $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ be a sublattice that is contained in $\mathcal{M}$ [3]. We have a constructible morphism

$$\underline{\pi}_X^{*B} := \underline{\pi}_X^* |_{\mathrm{Gr}_R(\mathcal{M})} : \ \mathrm{Gr}_R(\mathcal{M}) \xrightarrow{c} \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}). \tag{5.15}$$

We call this constructible morphism a **restricted lattice closure**. The graph *resp.* image of this morphism is a constructible ind-scheme that serves as a geometrization of the notion of restricted extension fiber *resp.* boundary as defined in §4.2.

We already see from Theorem 5.28 that when $\mathcal{M}$ is already a $\widetilde{R}$-lattice, then $\underline{\pi}_X^*$ is a constructible fibration of constant fibers $E_R(\widetilde{R})$. Unfortunately, this is not the case in general. The problem here is that $(\underline{\pi}_X^*)^{-1} \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}})$ is usually not constructibly identical to $\mathrm{Gr}_R(\mathcal{M})$, so one can not just restrict the fibration obtained from Theorem 5.28, as in the proof of Corollary 5.29. Our goal is to show that (5.15) is still a constructible fibration, but relative to a certain base. In order to achieve this, we will be using the following **restricted summation map**:

$$\mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}^B := \mathrm{Sum}_{\underset{\sim}{\mathcal{M}}} |_{\mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}})} : \ \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \xrightarrow{c} \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}). \tag{5.16}$$

Roughly speaking, our goal is to show that

$$\mathrm{Gr}_R(\mathcal{M}) \overset{c}{\simeq} \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \times_{\mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}^B, \, \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}})} (\mathrm{Fibers}). \tag{5.17}$$

5.5.1. *Restricted summation map.* In this particular paragraph we suppose that $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ is any pair of X-families of $\widetilde{R}$-lattices. We have

$$\mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \overset{c}{=} \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}) \times_{\mathrm{Gr}_{\widetilde{R}}([\underset{\sim}{\mathcal{M}}, \infty)), \, \mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}} \mathrm{Gr}_{\widetilde{R},X} \overset{c}{=} \mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}^{-1} \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}). \tag{5.18}$$

This can be easily checked over field valued points. Base change the fibrations (5.13) along the embedding $\mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}) \subseteq \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}})$ gives rise to fibration structures of $\mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}^B : \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \xrightarrow{c} \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}})$. We will abuse the notation, and still write $\mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}) \overset{c}{=} \bigsqcup_{\beta \in \mathbf{B}} Y_\beta$ for the corresponding stratification. Since $\mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}})$ is already a scheme, the index set $\mathbf{B}$ here is a finite set. Remark 5.33 also yields a loop space interpretation in this case: we can write

$$\left(\mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) \xrightarrow{\mathrm{Sum}_{\underset{\sim}{\mathcal{M}}}^B} \mathrm{Gr}_{\widetilde{R}}(\underset{\sim}{\mathcal{M}}, \widetilde{\mathcal{M}}) \right) \overset{c}{\simeq} \bigsqcup_{\beta \in \mathbf{B}} \left(\mathrm{Pd}_{\widetilde{R}}(p_2^* \underset{\sim}{\mathcal{M}}_{Y_\beta}, p_1^* \widetilde{\mathcal{U}}_{Y_\beta}) \to Y_\beta \right) \tag{5.19}$$

$$\simeq \bigsqcup_{\beta \in \mathbf{B}} \left(\mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta) \times Y_\beta \to Y_\beta \right) \tag{5.20}$$

$$\simeq \bigsqcup_{\beta \in \mathbf{B}} \left((L_+^K \mathcal{K}_{\mu_\beta} / L^{\widetilde{R}} \mathcal{K}_{\mu_\beta}) \times Y_\beta \to Y_\beta \right). \tag{5.21}$$

[3] Which always exists, see Remark 5.2. The reason for this somewhat technical assumption is to mimic Proposition 4.7.

5.5.2. *Restricted lattice closure as a relative fibration over restricted summation.* Recall that we assumed that $\widetilde{\mathcal{M}} = \widetilde{R}\mathcal{M}$ and let $\underset{\sim}{\mathcal{M}} \subseteq \widetilde{\mathcal{M}}$ be a sublattice that is contained in $\mathcal{M}$. We will also use the notation from §5.5.1. Let us fix one β. The following notation w be used:

- $\mathcal{P}_\beta := L_+^K\mathcal{K}_{\mu_\beta}/L^{\widetilde{R}}\mathcal{K}_{\mu_\beta} \simeq \mathrm{Pd}_{\widetilde{R}}(\underset{\sim}{N}_\beta, \widetilde{N}_\beta)$, $\mathscr{P}_\beta := \mathcal{P}_\beta \times Y_\beta \simeq \mathrm{Pd}_{\widetilde{R}}(p_2^*\underset{\sim}{\mathcal{M}}_{Y_\beta}, p_1^*\widetilde{\mathcal{U}}_{Y_\beta})$
- $\mathcal{P}_\beta^n := \mathrm{Pd}_{\widetilde{R}}^n(\underset{\sim}{N}_\beta, \widetilde{N}_\beta)$, $\mathscr{P}_\beta^n := \mathcal{P}_\beta^n \times Y_\beta \simeq \mathrm{Pd}_{\widetilde{R}}^n(p_2^*\underset{\sim}{\mathcal{M}}_{Y_\beta}, p_1^*\widetilde{\mathcal{U}}_{Y_\beta})$,
- $E_R(Y_\beta;\mathcal{M}) := (\underline{\pi}_X^{*B})^{-1}Y_\beta$, $E_R(\mathscr{P}_\beta;\mathcal{M}) := (\underline{\pi}_X^{*B})^{-1}\mathscr{P}_\beta$, they admit grading E_R^n from intersecting with $\mathrm{Gr}_R^n(\mathcal{M})$,
- $\mathrm{cl}_\beta := \underline{\pi}_X^{*B}|_{E_R(\mathscr{P}_\beta;\mathcal{M})}$. It is a constructible morphism relative to the base Y_β. We will write $\Gamma(\mathrm{cl}_\beta) \overset{c}{\subseteq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta$ for its relative graph.

Note that we have $E_R(\mathscr{P}_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta \overset{c}{\simeq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times \mathcal{P}_\beta$. The right hand side is an absolute product. Technically, it is easier and cleaner to work with the absolute version. So in the following, instead of working with $E_R(\mathscr{P}_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta$, we work with $E_R(\mathscr{P}_\beta;\mathcal{M}) \times \mathcal{P}_\beta$. In particular, we have $\Gamma(\mathrm{cl}_\beta) \overset{c}{\subseteq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times \mathcal{P}_\beta$. This means that we can also view cl_β as an absolute constructible morphism $E_R(\mathscr{P}_\beta;\mathcal{M}) \to \mathcal{P}_\beta$.

Remark 5.34. To reduce the level of abstractness, and for later use, we briefly describe the field valued points of several objects considered above. Let F be a field extension of k. By restricting to an F-point of X, we can assume that $X = \operatorname{Spec} F$. Then $\mathcal{M}$, $\widetilde{\mathcal{M}}$ and $\underset{\sim}{\mathcal{M}}$ are simply lattices over F. We have

$$E_R(\mathscr{P}_\beta;\mathcal{M})(F) = \{(L,\widetilde{L}) \in (\mathrm{Gr}_R(\mathcal{M}) \times Y_\beta)(F) | \widetilde{R}L + \underset{\sim}{\mathcal{M}} = \widetilde{L}\}. \tag{5.22}$$

The chain of maps $E_R(\mathscr{P}_\beta;\mathcal{M}) \to \mathscr{P}_\beta \to Y_\beta$, over F-points, can be described as $(L,\widetilde{L}) \to \widetilde{R}L \to \widetilde{L}$. Therefore, the points of $\Gamma(\mathrm{cl}_\beta) \overset{c}{\subseteq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta$, can be described as

$$\Gamma(\mathrm{cl}_\beta)(F) = \{(L,\widetilde{L},\widetilde{R}L) | (L,\widetilde{L}) \in E_R(\mathscr{P}_\beta;\mathcal{M})(F)\}. \tag{5.23}$$

On the other hand, if we view cl_β as a constructible morphism $E_R(\mathscr{P}_\beta;\mathcal{M}) \to \mathcal{P}_\beta$, then it sends $(L,\widetilde{L})$ to a point $\alpha \in \mathcal{P}_\beta(F)$, one (hence all) of whose lifts to $L_+^K\mathcal{K}_{\mu_\beta}(F)$ takes $\widetilde{L}$ to $\widetilde{R}L$. In this situation, we will write $\alpha\widetilde{L} = \widetilde{R}L$. Consequently, the points of $\Gamma(\mathrm{cl}_\beta) \overset{c}{\subseteq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times \mathcal{P}_\beta$ can be described as

$$\begin{aligned}\Gamma(\mathrm{cl}_\beta)(F) &= \{(L,\widetilde{L},\alpha) | (L,\widetilde{L}) \in E_R(\mathscr{P}_\beta;\mathcal{M})(F), \alpha \in \mathcal{P}_\beta(F), \alpha\widetilde{L} = \widetilde{R}L\} && (5.24)\\ &= \{(L,\widetilde{L},\alpha) | (\widetilde{L},\alpha) \in (Y_\beta \times \mathcal{P}_\beta)(F), L \in E_R(\alpha\widetilde{L};\mathcal{M})(F)\}. && (5.25)\end{aligned}$$

In (5.25), $E_R(\alpha\widetilde{L};\mathcal{M})$ is the restricted extension fiber of $\alpha\widetilde{L}$.

Theorem 5.35 (Restricted lattice closure as a relative fibration over restricted summation)**.** *Notation as above, we have*

$$E_R(\mathscr{P}_\beta;\mathcal{M}) \overset{c}{\simeq} E_R(Y_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta \overset{c}{\simeq} E_R(Y_\beta;\mathcal{M}) \times \mathcal{P}_\beta, \tag{5.26}$$

$$E_R^n(\mathscr{P}_\beta;\mathcal{M}) \overset{c}{\simeq} \bigsqcup_{i+j=n} E_R^i(Y_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta^j \overset{c}{\simeq} \bigsqcup_{i+j=n} E_R^i(Y_\beta;\mathcal{M}) \times \mathcal{P}_\beta^j \tag{5.27}$$

As a result, we have a relative constructible fibration

$$\begin{array}{ccccc} \mathrm{Gr}_R(\mathcal{M}) & \xrightarrow{\overset{c}{\sim}} & \bigsqcup_{\beta\in\mathbf{B}} E_R(Y_\beta;\mathcal{M}) \times_{Y_\beta} \mathscr{P}_\beta & \xrightarrow{\sim} & \bigsqcup_{\beta\in\mathbf{B}} E_R(Y_\beta;\mathcal{M}) \times \mathcal{P}_\beta \\ {\scriptstyle \underline{\pi}_X^{*B}}\downarrow & & \downarrow & & \downarrow \\ \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}}) & \xrightarrow{\overset{c}{\sim}} & \bigsqcup_{\beta\in\mathbf{B}} \mathscr{P}_\beta & = & \bigsqcup_{\beta\in\mathbf{B}} \mathscr{P}_\beta \end{array} \tag{5.28}$$

We recover Corollary 5.29 as a special case where $\underset{\sim}{\mathcal{M}} = \widetilde{\mathcal{M}}$, $\#\mathbf{B} = 1$, $\mathscr{P}_\beta = \mathrm{Gr}_{\widetilde{R}}(\widetilde{\mathcal{M}})$ and the structural map $Y_\beta \to X$ is an isomorphism.

Proof. It suffices to prove (5.26). The other assertions follow easily. The strategy is similar to that of Theorem 5.28: we use the loop group to permute the fibers. Since $E_R(\mathscr{P}_\beta;\mathcal{M}) \overset{c}{\simeq} \Gamma(\mathrm{cl}_\beta) \overset{c}{\subseteq} E_R(\mathscr{P}_\beta;\mathcal{M}) \times \mathcal{P}_\beta$, it suffices to show that the projection map $p_2 : \Gamma(\mathrm{cl}_\beta) \overset{c}{\to} \mathcal{P}_\beta$ is a constructible fibration with $\Gamma(\mathrm{cl}_\beta) \overset{c}{\simeq} E_R(Y_\beta;\mathcal{M}) \times \mathcal{P}_\beta$. Because $L_+^K\mathcal{K}_{\mu_\beta}$ is a Zariski $L^{\widetilde{R}}\mathcal{K}_{\mu_\beta}$-torsor over $\mathcal{P}_\beta$, $\mathcal{P}_\beta$ is covered by open affine subschemes W which admit sections $\sigma : W \to (L_+^K\mathcal{K}_{\mu_\beta})_W := L_+^K\mathcal{K}_{\mu_\beta} \times_{\mathcal{P}_\beta} W$. Let $\mathrm{cl}_{\beta,W}$ be the constructible morphism $\mathrm{cl}_\beta|_{\mathrm{cl}_\beta^{-1}(W)}$. It suffices to show that $\Gamma(\mathrm{cl}_{\beta,W}) \overset{c}{\simeq} E_R(Y_\beta;\mathcal{M}) \times W$.

We introduce an auxiliary ind-scheme $\mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) \subseteq \mathrm{Gr}_{R,Y_\beta}$. It has the following properties: (1) since $p_1^*\widetilde{\mathcal{U}}_{Y_\beta} \simeq \widetilde{N}_{\beta,Y_\beta}$, we have $\mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) \simeq \mathrm{Gr}_R(\widetilde{N}_\beta) \times Y_\beta$, (2) $E_R(\mathscr{P}_\beta;\mathcal{M}) \overset{c}{\subseteq} \mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta})$, and (3) $L_+^K\mathcal{K}_{\mu_\beta} \curvearrowright \mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta})$ by acting on the corresponding basis of $\widetilde{N}_\beta$ or $p_1^*\widetilde{\mathcal{U}}_{Y_\beta}$, see Remark 5.33. Similar to (5.12), we have an isomorphism:

$$\widetilde{\sigma} : \mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) \times W \to \mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) \times W,\ (\mathcal{L}, w) \to (\sigma(w)\mathcal{L}, w). \tag{5.29}$$

Let $U \overset{c}{\subseteq} \mathrm{Gr}_R(p_1^*\widetilde{\mathcal{U}}_{Y_\beta}) \times W$ be the image under $\widetilde{\sigma}$ of the constructible subscheme $E_R(Y_\beta;\mathcal{M}) \times W$. Then U is a constructible subscheme of $E_R(\mathscr{P}_\beta;\mathcal{M}) \times W$. Now we show that $U \overset{c}{=} \Gamma(\mathrm{cl}_{\beta,W})$ by checking the field valued points. Let F be a field extension of k. By restricting to an F-point of X, we can assume that $X = \operatorname{Spec} F$. The F-points of U can be described as

$$U(F) = \{(\sigma(w)N, \widetilde{L}, w) | (\widetilde{L}, w) \in (Y_\beta \times W)(F), N \in E_R(\widetilde{L};\mathcal{M})(F)\}. \tag{5.30}$$

On the other hand, (5.25) tells that

$$\Gamma(\mathrm{cl}_{\beta,W})(F) = \{(L, \widetilde{L}, w) | (\widetilde{L}, w) \in (Y_\beta \times W)(F), L \in E_R(\sigma(w)\widetilde{L};\mathcal{M})(F)\}. \tag{5.31}$$

Fix any pair $(\widetilde{L}, w)$ as in (5.30) and (5.31). The padding lemma (Lemma 4.3) implies that

$$E_R(\sigma(w)\widetilde{L};\mathcal{M})(F) = \sigma(w)E_R(\widetilde{L};\mathcal{M})(F). \tag{5.32}$$

This immediately shows that $U(F) = \Gamma(\mathrm{cl}_{\beta,W})(F)$. As a result $\Gamma(\mathrm{cl}_{\beta,W})(F) \overset{c}{=} U \overset{c}{\simeq} E_R(Y_\beta;\mathcal{M}) \times W$, and we are done. □

Remark 5.36. Even if one only cares about the absolute case (i.e., $X = \operatorname{Spec} k$), the fibration in Theorem 5.35 is still relative. For this reason, the relevant setting is inevitable.

5.6. **Rationality of the relative Quot zeta function.** We are ready to establish the rationality theorem. We will first prove a relative version and deduce the absolute version as a corollary. In the following, let X be a quasi-compact k-scheme and let $\mathcal{M}$ *resp.* $\widetilde{\mathcal{M}}$ be a rank d X-family of R-lattices *resp.* $\widetilde{R}$-lattices. We will be considering the relative Grothendieck ring of motives $K_0(\mathbf{Sch}_X)$. Its elements are morphisms of the form $[Y \to X]$. We abbreviate the projection $[X \times Z \to X]$ as $[Z]$. Following the above notation, we also write 1 *resp.* $\mathbb{L}$ for the motive $[\operatorname{Spec} k]$ *resp.* $[\mathbb{A}^1] \in K_0(\mathbf{Sch}_X)$. Recall the notion of the relative Quot scheme from Proposition 5.7. Define the relative Quote zeta function of $\mathcal{M}$ as

$$Z^R_{\mathcal{M}/X}(t) := \sum_{n\geq 0} [\mathrm{Quot}^R_{\mathcal{M}/X,n} \to X] t^n \in 1 + t \cdot K_0(\mathbf{Sch}_X)[[t]]. \tag{5.33}$$

By Proposition 5.7, we can replace $\mathrm{Quot}^R_{\mathcal{M}/X,n}$ in the above formula by $\mathrm{Gr}^n_R(\mathcal{M})$.

Example 5.37. It follows from Corollary 5.12 that $Z^{\widetilde{R}}_{\widetilde{\mathcal{M}}/X}(t) = (t;\mathbb{L})_d^{-s}$. It follows from Corollary 5.29 and Remark 5.23 that $Z^R_{\widetilde{\mathcal{M}}/X}(t) = (t;\mathbb{L})_d^{-s} \sum_{n\geq 0} [E^n_R(\widetilde{R}^d)] t^n$. The sum is indeed a polynomial since $[E^n_R(\widetilde{R}^d)] = 0$ for $n \gg 0$.

Theorem 5.38. *$Z^R_{\mathcal{M}/X}(t)$ is a rational function in t, i.e., it lies in $K_0(\mathbf{Sch}_X)(t)$. Moreover, the quotient $Z^R_{\mathcal{M}/X}(t)/Z^{\widetilde{R}}_{\widetilde{\mathcal{M}}/X}(t)$ is a polynomial in $K_0(\mathbf{Sch}_X)[t]$.*

Proof. We use the notation from §5.5. Note that $\widetilde{\mathcal{M}}/\mathcal{M}$ is of constant rank over X. We call the rank m. On the other hand, we can assume that Y_β is connected, then, as already noted in §5.1.1, the sheaf $(p_2^*\widetilde{\mathcal{M}}/p_1^*\widetilde{\mathcal{U}})_{Y_\beta}$ is of constant rank. We call the rank m_β. It follows from Theorem 5.35 that

$$\begin{aligned} Z^R_{\mathcal{M}/X}(t) &= \sum_{n\geq 0}[\mathrm{Gr}^n_R(\mathcal{M})\to X]t^n && (5.34)\\ &= \sum_{\beta\in\mathbf{B}} t^{m_\beta-m}\sum_{n\geq 0}[E^n(\mathscr{P}_\beta;\mathcal{M})\to X]t^n && (5.35)\\ &= \sum_{\beta\in\mathbf{B}} t^{m_\beta-m}\sum_{n\geq 0}\sum_{i+j=n}[E^i_R(Y_\beta;\mathcal{M})\to X][\mathcal{P}^j_\beta]t^n && (5.36)\\ &= \sum_{\beta\in\mathbf{B}} t^{m_\beta-m}\sum_{i\geq 0}[E^i_R(Y_\beta;\mathcal{M})\to X]t^i\sum_{j\geq 0}[\mathcal{P}^j_\beta]t^j && (5.37)\end{aligned}$$

It is easy to see that $E^i_R(Y_\beta;\mathcal{M})$ is empty for $i\gg 0$, so $E_\beta(t):=\sum_{i\geq 0}[E^i_R(Y_\beta;\mathcal{M})\to X]t^i$ is a polynomial in $K_0(\mathbf{Sch}_X)[t]$. On the other hand, by Remark 5.9, there exist $r_{\beta,i}$ such that

$$\sum_{j\geq 0}[\mathcal{P}^j_\beta]t^j = (t;\mathbb{L})_d^{-s}\prod_{i=1}^s (t;\mathbb{L})_{r_{\beta,i}}. \tag{5.38}$$

Combining with the computation of $Z^{\widetilde{R}}_{\widetilde{\mathcal{M}}/X}(t)$ in Example 5.37, we see that

$$Z^R_{\mathcal{M}/X}(t)/Z^{\widetilde{R}}_{\widetilde{\mathcal{M}}/X}(t) = \sum_{\beta\in\mathbf{B}} t^{m_\beta-m}E_\beta(t)\prod_{i=1}^s(t;\mathbb{L})_{r_{\beta,i}}. \tag{5.39}$$

Note that the right-hand side of (5.39) is t^{-m} times a polynomial. This already implies that $Z^R_{\mathcal{M}/X}(t)$ is rational. Furthermore, the left-hand side of (5.39) is $(t;\mathbb{L})_d^s\sum_{n\geq 0}[\mathrm{Gr}^n_R(\mathcal{M})\to X]t^n$, which only involves nonnegative power terms. This forces the right-hand side of (5.39) to be a polynomial. □

Corollary 5.39. *For a torsion free bundle E of rank d over R, the Quot zeta function $Z^R_E(t)$ is rational in t. Furthermore, $Z^R_E(t)/Z^{\widetilde{R}}_{\widetilde{R}^{\oplus d}}(t)$ is a polynomial in t.*

Proof. Let $X=\operatorname{Spec} k$, $\mathcal{M}=E$, and apply Theorem 5.38. □

Remark 5.40. The polynomiality is explicit. In (5.39), taking $X=\operatorname{Spec} k$, $\mathcal{M}=E$ and combining the strata β with the same $(m_\beta,(r_{\beta,i})_i)$, we get the following motivic upgrade of Proposition 4.7:

$$\frac{Z^R_E(t)}{Z^{\widetilde{R}}_{\widetilde{R}^{\oplus d}}(t)} = \sum_{n,r_1,\dots,r_s} t^n\left[\left\{(L,\widetilde{L})\,\middle|\, \begin{matrix}\widetilde{L}\supseteq_{\widetilde{R}} \underset{\sim}{M}, \mathrm{rk}_i(\widetilde{L}/\underset{\sim}{M})=r_i,\\ L\subseteq_R M, \widetilde{R}L=\widetilde{L}, [M:L]=n\end{matrix}\right\}\right]\prod_{i=1}^s(t;\mathbb{L})_{r_i}\in K_0(\mathrm{Var}_k)[t], \tag{5.40}$$

where the moduli space in the summand can be understood as a constructible subset in a certain flag Quot scheme.

6. Motivic Cohen–Lenstra zeta function

In this section, we state and prove the key geometric theorem that connects high-rank Quot zeta functions and the motivic Cohen–Lenstra zeta function. Assume (∗∗), i.e., let k be an arbitrary field and $(R,\mathfrak{m})$ be a complete local ring of the form $k[[x_1,\dots,x_m]]/I$ with residue field k. Let $\mathrm{Coh}^r_n(R)$ be the stack of R-modules of rank r (i.e., requiring r generators) and dimension n (over k). Denote $\mathrm{Quot}_{d,n}(R):=\mathrm{Quot}_{R^d,n}$ and let $\mathrm{Quot}^r_{d,n}(R)$ parametrize n-dimensional quotients of R^d of rank r. See §6.2 for precise definitions.

Theorem 6.1 ([37, Thm. B.1])**.** *Notation as above. For any n, d, r such that $0 \le r \le \min\{d, n\}$, the following identity holds in $K_0(\mathrm{Stck}_k)$:*

$$[\mathrm{Coh}_n^r(R)] = \frac{[\mathrm{Quot}_{d,n}^r(R)][\mathrm{GL}_{d-r}]}{\mathbb{L}^{d(n-d)+(d-r)^2}[\mathrm{GL}_d]} = \mathbb{L}^{-dn}\frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{d-r}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_d}[\mathrm{Quot}_{d,n}^r(R)]. \tag{6.1}$$

Note that the left-hand side does not depend on d. We will apply this theorem for both $d = r$ and $d > r$. Before proving Theorem 6.1, we prove Theorem 1.17 as a consequence.

Remark 6.2. The point-count version of Theorem 6.1 essentially follows from Nakayama's lemma. To sketch a proof, recall that when $k = \mathbb{F}_q$, the point count of $\mathrm{Coh}_n^r(R)$ is defined as a groupoid count $\sum_{M\in \mathrm{Coh}_n^r(R)} |\mathrm{Aut}(M)|^{-1}$, where $\mathrm{Coh}_n^r(R)$ now denotes the (finite) set of rank r, dimension n modules over R up to isomorphism. For each $M \in \mathrm{Coh}_n^r(R)$ and an integer $d \ge r$, consider the set $\mathrm{Surj}_R(R^d, M)$ of R-surjections $R^d \twoheadrightarrow M$ equipped with the natural $\mathrm{Aut}(M)$ action, which is free. Let $\mathrm{Quot}_{R^d,M}$ denote the orbit space $\mathrm{Surj}_R(R^d, M)/\mathrm{Aut}(M)$, then it naturally corresponds to the set consisting of quotients of R^d that are isomorphic to M. The point-count version of (6.1) then follows from $\mathrm{Quot}_{d,n}^r(R) = \bigsqcup_{M\in\mathrm{Coh}_n^r(R)} \mathrm{Quot}_{R^d,M}$, the evaluation of $|\mathrm{Surj}_R(R^d, M)|$ using Nakayama's lemma (which depends only on n, d, r but not on M), and summing up $|\mathrm{Surj}_R(R^d, M)|/|\mathrm{Aut}(M)|$ over $M \in \mathrm{Coh}_n^r(R)$. See a previous work of the authors for more details [37, Prop. 2.3].

6.1. **Proof of Theorem 1.12 assuming Theorem 6.1.** We first establish some conversion rules between $Z_{R^d}(t)$, $Z_{\mathfrak{m}R^d}(t)$, and $\widehat{Z}_R(t)$ that follow formally from Theorem 6.1. It is clear from the definition that $\mathrm{Coh}_n(R) = \bigsqcup_{r=0}^n \mathrm{Coh}_n^r(R)$, $\mathrm{Quot}_{d,n}(R) = \bigsqcup_{r=0}^{\min\{d,n\}} \mathrm{Quot}_{d,n}^r(R)$, and moreover, $\mathrm{Quot}_{r,n+r}^r(R) \simeq \mathrm{Quot}_{\mathfrak{m}R^r,n}$. Indeed, $L \subseteq_R R^r$ satisfies $\mathrm{rk}(R^r/L) = r$ if and only if $L \subseteq \mathfrak{m}R^r$ by Nakayama's lemma.

Lemma 6.3. *We have the following formal identities:*

$$Z_{R^d}^R(t) = \sum_{r=0}^d \begin{bmatrix} d \\ r \end{bmatrix}_{\mathbb{L}} t^r Z_{\mathfrak{m}R^r}^R(\mathbb{L}^{d-r}t) \in K_0(\mathrm{Stck}_k)[t], \tag{6.2}$$

$$t^d Z_{\mathfrak{m}R^d}^R(t) = (\mathbb{L}^{-1};\mathbb{L}^{-1})_d \sum_{r=0}^d \frac{\mathbb{L}^{d-r}}{(\mathbb{L};\mathbb{L})_{d-r}} \frac{1}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_r} Z_{R^r}^R(\mathbb{L}^{d-r}t) \in K_0(\mathrm{Stck}_k)[t], \tag{6.3}$$

$$\widehat{Z}_R(t) = \sum_{d=0}^\infty \frac{t^d}{\mathbb{L}^{d^2}(\mathbb{L}^{-1};\mathbb{L}^{-1})_d} Z_{\mathfrak{m}R^d}^R(\mathbb{L}^{-d}t) \in K_0(\mathrm{Stck}_k)[[t]], \tag{6.4}$$

$$\widehat{Z}_R(t) = \sum_{d=0}^\infty \sum_{r=0}^d \frac{\mathbb{L}^{d-r}}{(\mathbb{L};\mathbb{L})_{d-r}} \frac{1}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_r} Z_{R^r}(\mathbb{L}^{-r}t) \in K_0(\mathrm{Stck}_k)[[t]], \tag{6.5}$$

where the inner sum in (6.5) *is divisible by t^d.*

Proof. Since $[\mathrm{Quot}_{r,n+r}^r(R)] = [\mathrm{Quot}_{\mathfrak{m}R^r,n}]$, applying (6.1) with $d = r$ computes $[\mathrm{Coh}_{n+r}^r(R)]$ in terms of $[\mathrm{Quot}_{\mathfrak{m}R^r,n}]$. This leads to (6.4). Applying (6.1) again with $d \ge r$ computes $[\mathrm{Quot}_{d,n}^r(R)]$ in terms of $[\mathrm{Coh}_n^r(R)]$, and hence in terms of $\mathrm{Quot}_{\mathfrak{m}R^r,n-r}$. Summing $r = 0$ to d gives (6.2). Finally, (6.3) follows from (6.2) and q-Pascal inversion [61], and (6.5) follows from (6.4) and (6.3). See also [37, Cor. 2.7]. □

We are ready to prove Theorem 1.12. Recall that it has two parts: the $d \to \infty$ convergence of $Z_{R^d}(\mathbb{L}^{-d}t)$, and that its limit is $\widehat{Z}_R(t)$. To prove the first part, we will make use of (6.3). To prove the second part, we apply (6.5), but with an important caution that we cannot switch the order of the summations: letting $l = d - r$, the inner sum after the switch, which is $\sum_{l=0}^\infty \mathbb{L}^l/(\mathbb{L};\mathbb{L})_l$, converges to zero in $\mathbb{Z}[[\mathbb{L}^{-1}]]$ by Euler's identity, while $\widehat{Z}_R(t)$ is nonzero since the constant term is 1. Towards this issue, we will rewrite the inner sum of (6.5) first.

Proof of Theorem 1.12 assuming Theorem 6.1. For $d \geq 0$, define the following in $K_0(\mathrm{Stck}_k)[[t]]$ by letting $c_d(t) := Z_{R^d}(\mathbb{L}^{-d}t)/(\mathbb{L}^{-1};\mathbb{L}^{-1})_d$, $a_d(t) := c_d(t) - c_{d-1}(t)$ with $c_{-1}(t) := 0$, and $b_d(t)$ be the summand of (6.4). By (6.3) with $t \mapsto \mathbb{L}^{-d}t$, we have

$$\sum_{r=0}^{d} \frac{\mathbb{L}^{d-r}}{(\mathbb{L};\mathbb{L})_{d-r}} c_r(t) = b_d(t) \in t^d K_0(\mathrm{Stck}_k)[[t]]. \tag{6.6}$$

From the elementary identity

$$\frac{1}{(\mathbb{L};\mathbb{L})_{d-r}} - \frac{1}{(\mathbb{L};\mathbb{L})_{d-r-1}} = \frac{\mathbb{L}^{d-r}}{(\mathbb{L};\mathbb{L})_{d-r}}, \quad 0 \leq r \leq d-1, \tag{6.7}$$

it follows that

$$b_d(t) = \sum_{r=0}^{d} \frac{\mathbb{L}^{d-r}}{(\mathbb{L};\mathbb{L})_{d-r}} c_r(t) = \sum_{r=0}^{d} \frac{1}{(\mathbb{L};\mathbb{L})_{d-r}} a_r(t). \tag{6.8}$$

Recall that the topology on $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]][[t]]$ is t-coefficientwise. We now show that each coefficient of $a_d(t)$ converges in $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$. Fix $N \geq 0$, and let $a_{d,N}$ denote the t^N-coefficient of $a_d(t)$. Since $b_d(t) \in t^d K_0(\mathrm{Stck}_k)[[t]]$, Equation (6.8) implies the following recurrence relation for $d \geq N+1$:

$$a_{d,N} = -\sum_{r=0}^{d-1} \frac{1}{(\mathbb{L};\mathbb{L})_{d-r}} a_{r,N}. \tag{6.9}$$

By setting $a_{d,N} = 0$ for $d < 0$, we may rewrite (6.9) into an $\infty \times \infty$ matrix form

$$v_d = M v_{d-1}, \quad d \geq N+1, \tag{6.10}$$

where

$$M = -\begin{bmatrix} \frac{1}{(\mathbb{L};\mathbb{L})_1} & \frac{1}{(\mathbb{L};\mathbb{L})_2} & \frac{1}{(\mathbb{L};\mathbb{L})_3} & \cdots \\ 0 & \frac{1}{(\mathbb{L};\mathbb{L})_1} & \frac{1}{(\mathbb{L};\mathbb{L})_2} & \cdots \\ 0 & 0 & \frac{1}{(\mathbb{L};\mathbb{L})_1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad v_d = \begin{bmatrix} a_{d,N} \\ a_{d-1,N} \\ a_{d-2,N} \\ \vdots \end{bmatrix}. \tag{6.11}$$

(Note that every entry in the matrix multiplication is just a finite sum.) It follows that $v_d = M^{d-N} v_N$ for all $d \geq N$. Observing that every entry of M has *dimension* at most -1 in $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$ in the sense of the dimension filtration, it follows that every entry of M^{d-N} has dimension at most $-(d-N)$. Therefore, every entry of v_d converges to 0. In particular, $\lim_{d\to\infty} a_{d,N} = 0$, so that $\lim_{d\to\infty} a_d(t) = 0$.

By definition, the topology on $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]][[t]]$ is nonarchimedean in the sense that for any sequence $x_d(t)$, the sum $\sum_d x_d(t)$ converges if and only if $\lim_{d\to\infty} x_d(t) = 0$, and in this case any reordering of the sum converges to the same limit. Therefore in our case, the sum $c(t) := \sum_{d=0}^{\infty} a_d(t)$ converges. By the definitions of $c_d(t)$ and $a_d(t)$, we have

$$\lim_{d\to\infty} Z_{R^d}(\mathbb{L}^{-d}t) = (\mathbb{L}^{-1};\mathbb{L}^{-1})_\infty \lim_{d\to\infty} c_d(t) = (\mathbb{L}^{-1};\mathbb{L}^{-1})_\infty c(t) \in K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}, t]], \tag{6.12}$$

proving the first assertion of Theorem 1.12.

Finally, (6.8) and the definition of $b_d(t)$ imply

$$\widehat{Z}_R(t) = \sum_{d=0}^{\infty} \sum_{r=0}^{d} \frac{1}{(\mathbb{L};\mathbb{L})_{d-r}} a_r(t). \tag{6.13}$$

By $\lim_{d\to\infty} a_d(t) = 0$ and the nonarchimedean property, we may switch the summations:

$$\widehat{Z}_R(t) = \sum_{r=0}^{\infty} a_r(t) \sum_{l=0}^{\infty} \frac{1}{(\mathbb{L};\mathbb{L})_l} = c(t)(\mathbb{L}^{-1};\mathbb{L}^{-1})_\infty. \tag{6.14}$$

Comparing with (6.12), the last assertion of Theorem 1.12 then follows. $\square$

Remark 6.4. The same argument proves the following analytic analogue of Theorem 1.12: when $k = \mathbb{F}_q$ is a fixed finite field with $q > 2$, and assume $(**)$ for R, then every t-coefficient of $|Z_{R^d}(\mathbb{L}^{-d}t)|_q$ converges as $d \to \infty$, and we have

$$|\widehat{Z}_R(t)|_q = \lim_{d\to\infty} |Z_{R^d}(\mathbb{L}^{-d}t)|_q \in \mathbb{C}[[t]] \tag{6.15}$$

coefficientwise. Indeed, we only need to note that the matrix M in (6.11) with $\mathbb{L}$ replaced by q satisfies that every entry of the matrix power M^n has absolute value bounded above by $C(q-1.01)^{-n}$ for some absolute constant C.

6.2. **Commuting varieties.** We now devote the rest of the section to proving Theorem 6.1 in the Grothendieck ring of algebraic k-stacks (for an introduction, see [20]). We first review the functor-of-points definitions of the stack $\mathrm{Coh}_n(R)$ and the Quot scheme $\mathrm{Quot}_{d,n}(R) := \mathrm{Quot}_{R^d,n}$, and clarify their relations with the commuting variety $C_n(R)$ also defined below. Fix a vector space V_n of dimension n over k. Consider the (sheafifications of the) following functors, where X is an arbitrary k-scheme:

$$\mathrm{Coh}_n(R)(X) = \begin{Bmatrix} \text{groupoid of coherent sheaves } \mathcal{F} \text{ over } X_R, \\ \text{such that } \mathcal{F} \text{ is flat of rank } n \text{ over } X \end{Bmatrix}, \tag{6.16}$$

$$\mathrm{Quot}_{d,n}(R)(X) = \left\{ \mathcal{O}_{X_R}^d \twoheadrightarrow \mathcal{F}, \text{ such that } \mathcal{F} \in \mathrm{Coh}_n(R)(X) \right\}, \tag{6.17}$$

$$C_n(R)(X) = \begin{Bmatrix} \text{equivalent classes of pairs } (\mathcal{F}, \iota), \text{ where } \mathcal{F} \in \mathrm{Coh}_n(R)(X), \\ \text{and } \iota : \mathcal{O}_X \otimes V_n \simeq \mathcal{F} \text{ is an isomorphism of } \mathcal{O}_X\text{-modules} \end{Bmatrix}. \tag{6.18}$$

It is well known that $C_n(R)$ is represented by a scheme parametrizing certain commuting matrices, and $\mathrm{Coh}_n(R)$ is the quotient stack $[C_n(R)/\mathrm{GL}_n]$; see e.g. [9, 36]. In fact, if we write $R = k[[T_1, \dots, T_m]]/(f_1, \dots, f_r)$, then $C_n(R)$ is the subscheme of $\mathrm{Nilp}_n(k)^m$ cut out by the condition $\underline{A} = (A_1, \dots, A_m) \in \mathrm{Nilp}_n(k)^m : [A_i, A_j] = 0, f_i(\underline{A}) = 0$. Here $\mathrm{Nilp}_n(k)$ is the variety of $n \times n$ nilpotent matrices over k.

Let $\mathcal{F}_{\mathrm{uv}}$ be the universal quotient sheaf over $\mathrm{Quot}_{d,n}(R)_R$. It can be viewed as a sheaf of R-module over $\mathrm{Quot}_{d,n}(R)$. Using a flattening stratification on $\mathrm{Quot}_{d,n}(R)$ corresponding to the finite R-module $\mathcal{F}_{\mathrm{uv}}/\mathfrak{m}\mathcal{F}_{\mathrm{uv}}$, we get $\mathrm{Quot}_{d,n}(R) \overset{c}{=} \bigsqcup \mathrm{Quot}_{d,n}^r(R)$ and each $\mathrm{Quot}_{d,n}^r(R)$ admits a functor of points

$$\mathrm{Quot}_{d,n}^r(R)(X) = \left\{ \mathcal{O}_{X_R}^d \twoheadrightarrow \mathcal{F}, \text{ such that } \mathcal{F} \in \mathrm{Coh}_n(R)(X) \text{ and } \mathcal{F}/\mathfrak{m}\mathcal{F} \in \mathrm{Coh}_r(R)(X) \right\}. \tag{6.19}$$

The condition that $\mathcal{F}/\mathfrak{m}\mathcal{F} \in \mathrm{Coh}_r(R)(X)$ can be understood as $\mathcal{F}$ has constant R-rank r. Note that $\mathrm{Quot}_{d,n}^r(R)$ is empty once $r > \min\{d, n\}$.

Similarly, we can put a flattening stratification $C_n(R) = \bigsqcup C_n^r(R)$ such that

$$C_n^r(R)(X) = \begin{Bmatrix} \text{equivalent classes of pairs } (\mathcal{F}, \iota) \text{ where } \mathcal{F} \in \mathrm{Coh}_n(R)(X), \mathcal{F}/\mathfrak{m}\mathcal{F} \in \mathrm{Coh}_r(R)(X) \\ \text{and } \iota : \mathcal{O}_X \otimes V_n \simeq \mathcal{F} \text{ is an isomorphism of } \mathcal{O}_X\text{-modules} \end{Bmatrix}. \tag{6.20}$$

Again, $C_n^r(R)$ is empty once $r > n$. We then set $\mathrm{Coh}_n^r(R) \overset{c}{=} [C_n^r(R)/\mathrm{GL}_n]$, and this induces a stratification $\mathrm{Coh}_n(R) \overset{c}{=} \bigsqcup \mathrm{Coh}_n^r(R)$.

Remark 6.5. When n is fixed, there is a large Artin local quotient R' of R such that for all d and r, we have that $\mathrm{Coh}_n(R)$, $\mathrm{Coh}_n^r(R)$, $\mathrm{Quot}_{d,n}(R)$, $\mathrm{Quot}_{d,n}^r(R)$, $C_n(R)$ and $C_n^r(R)$ are equal to $\mathrm{Coh}_n(R')$, $\mathrm{Coh}_n^r(R')$, $\mathrm{Quot}_{d,n}(R')$, $\mathrm{Quot}_{d,n}^r(R')$, $C_n(R')$ and $C_n^r(R')$, respectively.

6.3. **Proof of Theorem 6.1.** In the following, we will always assume that S is a k-scheme. We will fix n and r. By Remark 6.5, we may and do replace R by a large Artin quotient. In what follows, we will write $\dim_k R = b + 1$ and $F = R^d = \bigoplus_{i=1}^d Ru_i$.

6.3.1. *Proof outline of Theorem 6.1.* In §6.3.4, we will first observe a group action $\mathrm{GL}[F] \curvearrowright \mathrm{Quot}^r_{d,n}(R)$ and introduce an important group scheme $\mathscr{G}$ over $\mathrm{Quot}^r_{d,n}(R)$. In §6.3, we will construct a Zariski GL_n-torsor $\mathfrak{Q}$ over $\mathrm{Quot}^r_{d,n}(R)$ that sits in the following diagram

$$\mathrm{Quot}^r_{d,n}(R) \xleftarrow{\ \pi\ } \mathfrak{Q} \xrightarrow{\ \varpi\ } C^r_n(R) \tag{6.21}$$

We will show that ϖ admits Zariski local sections, i.e., we have a Zariski cover $U \to C^r_n(R)$, and $\sigma : U \to \mathfrak{Q}_U := \mathfrak{Q} \times_{C^r_n(R)} U$ such that $\varpi_U \sigma = \mathrm{Id}_U$. We then show that $\mathrm{GL}[F]_U$ is an fppf $\mathscr{G}_U$-torsor over $\mathfrak{Q}_U$, where $\mathscr{G}_U = \sigma^*\pi^*\mathscr{G}$ is a group scheme over U. We then show in Lemma 6.10 that U admits a Zariski stratification over which $\mathscr{G}_U$ is special (in the sense that any fppf torsor is Zariski locally trivial). We then use explicit descriptions of $\mathrm{GL}[F]$ and $\mathscr{G}$ from Lemma 6.6 and Lemma 6.7 to establish the formula (6.1).

6.3.2. *Vector bundles over* $\mathrm{Quot}^r_{d,n}(R)$. The terminology "vector bundle" is used for either a flat (locally free) sheaf of constant rank, or its total space (which is a scheme). Consider the universal sequence of R-modules over $\mathrm{Quot}^r_{d,n}(R)$:

$$0 \to \mathscr{N} \to \underline{F} \to \mathscr{Q} \to 0, \tag{6.22}$$

where $\underline{F}$ is the trivial bundle $F \otimes_k \mathcal{O}_{\mathrm{Quot}^r_{d,n}(R)}$, $\mathcal{Q}$ is the universal quotient, and $\mathscr{N}$ is the kernel. Forgetting the R-module structure, then $\mathscr{N}, \underline{F}$ and $\mathscr{Q}$ are vector bundles over $\mathrm{Quot}^r_{d,n}(R)$ of rank $d(b+1)-n$, $d(b+1)$ and n, respectively. Modulo $\mathfrak{m}$, we get another sequence

$$0 \to \overline{\mathscr{N}} \to \underline{\overline{F}} \to \overline{\mathscr{Q}} \to 0, \tag{6.23}$$

where $\overline{F} = F/\mathfrak{m}F$, $\underline{\overline{F}}$ is the constant bundle with fibers in $\overline{F}$, $\overline{\mathscr{Q}} = \mathscr{Q}/\mathfrak{m}\mathscr{Q}$, and $\overline{\mathscr{N}} = \mathscr{N}/(\mathscr{N} \cap \mathfrak{m}\underline{F})$. Again, $\overline{\mathscr{N}}, \underline{\overline{F}}$ and $\overline{\mathscr{Q}}$ are vector bundles over $\mathrm{Quot}^r_{d,n}(R)$ of rank $d-r$, d and r, respectively. Since $\mathscr{N}$ and $\overline{\mathscr{N}}$ are vector bundles, it also follows that $\mathscr{N} \cap \mathfrak{m}\underline{F}$ is a vector bundle. Its rank is $db - n + r$.

6.3.3. *Group action on* $\mathrm{Quot}^r_{d,n}(R)$. For a finite R-module M, let $\mathrm{GL}[M]$ be the group $\mathrm{Aut}_R(M)$, but viewed as a linear algebraic group over k. Then the group $\mathrm{GL}[F]$ acts on both $\mathrm{Quot}^r_{d,n}(R)$ and the bundle $\mathscr{Q}$ in a compatible way. In fact, let x be an S-point of $\mathrm{Quot}^r_{d,n}(R)$ corresponding to $F \otimes S \twoheadrightarrow (F \otimes S)/N$, and let g be an S-point of $\mathrm{GL}[F]$. Then gx is the point corresponding to $F \otimes S \twoheadrightarrow (F \otimes S)/g \cdot N$. Furthermore, the g action on $\mathscr{Q}$ takes the fiber $\mathscr{Q}_x$ to $\mathscr{Q}_{gx}$, and can be described as $(F \otimes S)/N \xrightarrow{g} (F \otimes S)/gN$.

There is also a projection $\mathrm{GL}[F] \to \mathrm{GL}[\overline{F}]$ sending g to $(g \bmod \mathfrak{m})$. Let $U(\mathfrak{m}F)$ be its kernel. The structure of $\mathrm{GL}[F]$ is easy to understand:

Lemma 6.6. *Notation as above, the exact sequence*

$$1 \to U[\mathfrak{m}F] \to \mathrm{GL}[F] \to \mathrm{GL}[\overline{F}] \to 1, \tag{6.24}$$

admits a canonical splitting. Furthermore, $\mathrm{GL}[\overline{F}] \simeq \mathrm{GL}_d$ *and* $U[\mathfrak{m}F]$ *is a k-split unipotent group of dimension* bd^2 *(a unipotent group is k-split if it is a successive extension of* $\mathbb{G}_a$ *over k).*

Proof. It is clear that $\mathrm{GL}[\overline{F}] \simeq \mathrm{GL}_d$. The morphism $\pi : \mathrm{GL}[F] \to \mathrm{GL}[\overline{F}]$ has a section $\mathrm{GL}[\overline{F}] \hookrightarrow \mathrm{GL}[F]$ induced by the structural morphism $k \hookrightarrow R$. Note that $U[\mathfrak{m}F](S)$ consists of g of form

$$g(u_i) \in u_i + \mathfrak{m}F \otimes S,\ 1 \le i \le d. \tag{6.25}$$

Therefore, $U[\mathfrak{m}F]$ is abstractly isomorphic to the affine space $\mathbb{A}^{bd^2}$. It is unipotent since for any $g \in U[\mathfrak{m}F](S)$, $(g - \mathrm{Id})^b = 0$. The fact that $U[\mathfrak{m}F]$ is k-split follows by induction on b (the k-dimension of $\mathfrak{m}$). The assertion is trivial for $b = 1$. Assuming otherwise, there is an element $0 \neq v \in \mathfrak{m}$ such that

$v\mathfrak{m}=0$. The subgroup $U[vF]\subseteq U[\mathfrak{m}F]$ whose S-points are elements g such that $g(u_i)\in u_i+vF\otimes S$ lies in the center of $U[\mathfrak{m}F]$, and is isomorphic to $\mathbb{G}_a^{d^2}$. Let $F'=(R/v)^d$. We have an exact sequence

$$1\to U[vF]\to U[\mathfrak{m}F]\to U[\mathfrak{m}F']\to 1. \tag{6.26}$$

By the induction hypothesis, $U[\mathfrak{m}F']$ is k-split. It follows that $U[\mathfrak{m}F]$ is k-split. □

6.3.4. *Group schemes over* $\mathrm{Quot}^r_{d,n}(R)$. Let $\mathrm{GL}[\underline{F}]$, $\mathrm{GL}[\overline{\underline{F}}]$ and $U[\mathfrak{m}\underline{F}]$ be constant group schemes over $\mathrm{Quot}^r_{d,n}(R)$ with fibers $\mathrm{GL}[F]$, $\mathrm{GL}[\overline{F}]$ and $U[\mathfrak{m}F]$, respectively. Note that $\mathrm{GL}[\underline{F}]$ acts on $\underline{F}$ in a tautological way. The readers shall keep in mind that all group scheme actions in this paragraph are relative over the base $\mathrm{Quot}^r_{d,n}(R)$, and should not be confused with the ones introduced in §6.3.3. Now we define the following:

- $\mathscr{G}$ is the subgroup scheme of $\mathrm{GL}[\underline{F}]$ stabilizing $\mathscr{N}$ and acts trivially on $\mathscr{Q}$.
- $\mathrm{Fil}_1\mathscr{G}$ is the subgroup scheme of $\mathscr{G}$ that acts trivially on $\overline{\underline{F}}$. It is equal to $U(\mathfrak{m}\underline{F})\cap\mathscr{G}$.
- $\mathrm{Fil}_2\mathscr{G}$ is the subgroup scheme of $\mathscr{G}$ that acts as trivially on both $\overline{\mathscr{N}}$ and $\overline{\mathscr{Q}}$.
- $\mathrm{GL}[\overline{\mathscr{N}}]:=\mathrm{Aut}_R(\overline{\mathscr{N}})$.

The group scheme $\mathscr{G}$ admits a three-step filtration of normal subgroups:

$$\underline{1}\lhd\mathrm{Fil}_1\mathscr{G}\lhd\mathrm{Fil}_2\mathscr{G}\lhd\mathrm{Fil}_3\mathscr{G}:=\mathscr{G}. \tag{6.27}$$

The graded objects $\mathrm{Fil}_i\mathscr{G}/\mathrm{Fil}_{i-1}\mathscr{G}$ are denoted by $\mathrm{gr}_i\mathscr{G}$.

Lemma 6.7. *The following are true:*

(a) *The underlying scheme of* $\mathrm{Fil}_1\mathscr{G}$ *is a vector bundle of rank* $d(bd-n+r)$.
(b) $\mathrm{gr}_2\mathscr{G}$ *is a subgroup scheme of* $\mathrm{GL}[\overline{\underline{F}}]$ *which is Zariski locally isomorphic to* $\mathbb{G}_a^{r(d-r)}$.[4]
(c) *The underlying scheme of* $\mathrm{Fil}_2\mathscr{G}$ *is a vector bundle of rank* $bd^2-d(n-d)-(d-r)^2$.
(d) $\mathrm{gr}_3\mathscr{G}\simeq\mathrm{GL}[\overline{\mathcal{N}}]$. *In particular, it is Zariski locally isomorphic to* GL_{d-r}.

Proof.

(a) Let $\mathrm{Spec}\,S$ be a Zariski affine cover of $\mathrm{Quot}^r_{d,n}(R)$. Then

$$\mathrm{Fil}_1\mathscr{G}(S)=\{g\in\mathrm{GL}[F](S)\,|\,g(u_i)\in u_i+H^0(\mathrm{Spec}\,S,\mathscr{N}\cap\mathfrak{m}\underline{F}),1\le i\le d\}. \tag{6.28}$$

This implies that as a scheme, $\mathrm{Fil}_1\mathscr{G}$ is isomorphic to the total space of $(\mathscr{N}\cap\mathfrak{m}\underline{F})^{\oplus d}$, which is a vector bundle of rank $d(bd-n+r)$.

(b) Let $\mathscr{H}_2$ be the subgroup scheme of $\mathrm{GL}[\overline{\underline{F}}]$ fixing (6.23) and acting trivially on both $\overline{\mathscr{Q}}$ and $\overline{\mathscr{N}}$. Then $\mathrm{gr}_2\mathscr{G}\subseteq\mathscr{H}_2$. The canonical splitting of (6.24) gives rise to a morphism

$$\mathscr{H}_2\hookrightarrow\mathrm{GL}[\overline{\underline{F}}]\to\mathrm{GL}[\underline{F}] \tag{6.29}$$

whose image lies in $\mathrm{Fil}_2\mathscr{G}$. This implies that $\mathrm{gr}_2\mathscr{G}=\mathscr{H}_2$. Let $\mathrm{Spec}\,S$ be a Zariksi affine open of $\mathrm{Quot}^r_{d,n}(R)$ over which $\overline{\mathscr{Q}}$ and $\overline{\mathscr{N}}$ are trivialized. We can then pick a section of (6.23) and write $\overline{\underline{F}}_S=\overline{\mathscr{N}}_S\oplus\overline{\mathscr{Q}}_S$. Then the S-points of $\mathscr{H}_2|_S$ can be written as matrices

$$\begin{bmatrix}\mathrm{Id}_{\overline{\mathscr{N}}_S} & B\\ 0 & \mathrm{Id}_{\overline{\mathscr{Q}}_S}\end{bmatrix},\ B\in\mathcal{H}om(\overline{\mathscr{Q}},\overline{\mathscr{N}})(S). \tag{6.30}$$

This particularly implies that $\mathrm{gr}_2\mathscr{G}|_S=\mathbb{G}_{a,S}^{r(d-r)}$.

(c) This follows from (a) and (b).

(d) Let $\mathscr{H}$ be the subgroup scheme of $\mathrm{GL}[\overline{\underline{F}}]$ fixing the filtration (6.23) and reducing to the identity on $\overline{\mathscr{Q}}$. We have $\mathscr{G}/\mathrm{Fil}_1\mathscr{G}\subseteq\mathscr{H}$. As in (b), the canonical splitting of (6.24) again induces a morphism $\mathscr{H}\to\mathrm{GL}[\underline{F}]$ whose image lies in $\mathscr{G}$. This implies that $\mathscr{G}/\mathrm{Fil}_1\mathscr{G}=\mathscr{H}$ and $\mathrm{gr}_3\mathscr{G}=\mathscr{H}/\mathscr{H}_2$.

[4] By this we mean that there is a Zariski cover U of $\mathrm{Quot}^r_{d,n}(R)$ such that $\mathrm{gr}_2\mathscr{G}_U\simeq\mathbb{G}_{a,U}^{r(d-r)}$ as group schemes. We adopt the same convention for (d).

Let $\operatorname{Spec} S$ be a Zariski affine open of $\mathrm{Quot}^r_{d,n}(R)$ over which $\overline{\mathscr{Q}}$ and $\overline{\mathscr{N}}$ are trivialized. Pick the same splitting of (6.23) as in (b). Then the S-points of $\mathscr{H}|_S$ can be written as matrices

$$\begin{bmatrix} A & B \\ 0 & \mathrm{Id}_{\overline{\mathscr{Q}}_S} \end{bmatrix}, \ A \in \mathrm{GL}[\overline{\mathscr{N}}](S), \ B \in \mathcal{H}om(\overline{\mathscr{Q}}, \overline{\mathscr{N}})(S). \tag{6.31}$$

Taking its quotient by by (6.30), the assertion that $\mathrm{gr}_3\,\mathscr{G} \simeq \mathrm{GL}[\overline{\mathcal{N}}]$ follows. □

6.3.5. *The torsor.* We start by constructing $\mathfrak{Q}$ in (6.21) as the GL_n-torsor corresponding to the bundle $\mathscr{Q}$. Recall that V_n is a fixed k-vector space of dimension n, and let $\underline{V_n} := V_n \otimes_k \mathcal{O}_{\mathrm{Quot}^r_{d,n}(R)}$ be the corresponding trivial bundle, then

$$\mathfrak{Q} = \mathcal{I}som(\underline{V_n}, \mathscr{Q}). \tag{6.32}$$

We have a projection $\pi : \mathfrak{Q} \to \mathrm{Quot}^r_{d,n}(R)$. Note that GL_n acts on $\mathfrak{Q}$ via its action on $\underline{V_n}$, while $\mathrm{GL}[F]$ acts on $\mathfrak{Q}$ via its action on $\mathscr{Q}$.

Since $\pi^*\mathfrak{Q} \to \mathfrak{Q}$ is a trivial torsor (e.g., there is a global section which comes from the diagonal), we see that $\pi^*\mathscr{Q} \to \mathfrak{Q}$ is a trivial bundle. It is an object in $\mathrm{Coh}^r_n(R)(\mathfrak{Q})$. Let

$$\iota : \mathcal{O}_{\mathfrak{Q}} \otimes_k V_n \simeq \pi^*\mathscr{Q} \tag{6.33}$$

be the isomorphism of $\mathcal{O}_{\mathfrak{Q}}$-modules induced by (6.32). The pair $(\pi^*\mathscr{Q}, \iota)$ gives rise to a morphism $\varpi : \mathfrak{Q} \to C^r_n(R)$ via the functor of points description (6.20).

On the level of S-points, ϖ can be easily understood as follows. Let $\mathfrak{Q}$ be trivialized over a point $x_0 : \operatorname{Spec} S \to \mathrm{Quot}^r_{d,n}(R)$, and let $x \in \mathfrak{Q}(S)$ be such that $\pi(x) = x_0$. Then x corresponds to an S-isomorphism $\iota : V_n \otimes S \simeq (F \otimes S)/N$. We then have $\varpi(x) = ((F \otimes S)/N, \iota)$. It is easily seen that ϖ is surjective, and is invariant under the $\mathrm{GL}[F]$-action. Indeed, let $g \in \mathrm{GL}[F](S)$ and let x be as above, then we have $\varpi g(x) = ((F \otimes S)/gN, g\iota) \sim ((F \otimes S)/N, \iota)$. This implies that $\varpi g(x) = \varpi(x)$.

6.3.6. *Local sections.* There is a Zariski affine cover $U = \operatorname{Spec} D \to C^r_n(R)$, such that the tautological point of $C^r_n(R)(U)$ corresponds to (the equivalent class of) a pair

$$((F \otimes D)/N^{\mathrm{taut}}, \iota^{\mathrm{taut}}). \tag{6.34}$$

Fixing a pair (6.34) in the equivalence class, we get a U-point σ of $\mathfrak{Q}$ such that $\varpi\sigma = \mathrm{Id}_U$. Therefore, σ is a section of ϖ over U. Two such sections differ by an action of $\mathrm{GL}[F](U)$. Let $\mathfrak{Q}_U = \mathfrak{Q} \times_{C^r_n(R)} U$. Since ϖ is invariant under $\mathrm{GL}[F]$, the section σ induces a morphism of U-schemes

$$p_\sigma : \mathrm{GL}[F]_U = U \times \mathrm{GL}[F] \xrightarrow{\sigma \times \mathrm{Id}} \mathfrak{Q} \times \mathrm{GL}[F] \xrightarrow{\cdot} \mathfrak{Q}. \tag{6.35}$$

Lemma 6.8. *Let $\mathscr{G}$ be as in §6.3.4 and let $\mathscr{G}_U := \sigma^*\pi^*\mathscr{G}$ be the pullback group scheme over U. Then p_σ realizes $\mathfrak{Q}_U$ as the fppf quotient of $\mathrm{GL}[F]_U$ by the fppf subgroup $\mathscr{G}_U$. In particular, we have:*

(a) *p_σ is an fppf cover.*
(b) *$\mathrm{GL}[F]_U$ is an fppf $\mathscr{G}_U$-torsor over $\mathfrak{Q}_U$ trivialized by p_σ.*

Proof. Let x be an S-point of U such that $\sigma(x)$ corresponds to an S-isomorphism $\iota : V_n \otimes S \simeq (F \otimes S)/N$, so x itself corresponds to $((F \otimes S)/N, \iota)$. For an S-point (x, g) of $U \times \mathrm{GL}[F]$, $p_\sigma(x, g)$ corresponds to the isomorphism $g\iota : V_n \otimes S \simeq (F \otimes S)/gN$. It is clear that p_σ surjects onto $\mathfrak{Q}_U$. On the other hand, $p_\sigma(x, g) = p_\sigma(x, g')$ if and only if $g'g^{-1}$ stablizes N and acts trivially on the quotient $(F \otimes S)/N$. This means that $g'g^{-1}$ is an S-point of $\mathscr{G}_U$ that lies above x, so we obtain the fppf quotient assertion. Once this is established, (a) and (b) are just formal properties of fppf quotients, see [18, Prop. 4.31]. □

6.3.7. *The proof.* We say that a unipotent group scheme over a base Y is Y-**split**, or simply **split**, if it is a successive extension of $\mathbb{G}_{a,Y}$. We say a group scheme $\mathcal{G}$ over Y is **special**, if any fppf $\mathcal{G}$-torsor is Zariski locally trivial. It is well-known that a split unipotent group scheme is special, and an extension of special group schemes is special.

Lemma 6.9. *Notation as in §6.3.4. There is a Zariski stratification*

$$\operatorname{Quot}^r_{d,n}(R) \overset{c}{=} \bigsqcup_{\beta\in\mathbf{B}} Z_\beta, \tag{6.36}$$

such that for each β, $\operatorname{Fil}_2 \mathscr{G}_{Z_\beta}$ is a Z_β-split unipotent group scheme.

Proof. Note that it suffices to show that $\operatorname{Fil}_1 \mathscr{G}_{Z_\beta}$ is split for some Zariski stratification $\{Z_\beta\}$, thanks to Lemma 6.7(b). Take a nonempty affine open $\operatorname{Spec} S \subseteq \operatorname{Quot}^r_{d,n}(R)$ and view it as an S-point of $\operatorname{Quot}^r_{d,n}(R)$. It then corresponds to a submodule $N \subseteq F \otimes S$. Shrinking S, we can assume that $N' = N \cap (\mathfrak{m}F \otimes S)$ is free. The group scheme $\operatorname{Fil}_1 \mathscr{G}_S$ can be described as the functor

$$T \in \mathbf{Alg}_S \mapsto \{g \in U[\mathfrak{m}F](T) : g(u_i) \in u_i + N' \otimes T, 1 \le i \le d\}. \tag{6.37}$$

Let $\{\omega_i\}_{i=1}^{bd+d-n}$ be a basis of N'. There is a nonzero element $a \in N'$ such that $\mathfrak{m}a = 0$. Indeed, we know that for any element $b \in N'$, $\mathfrak{m}^n b = 0$ for sufficiently large n. So multiply b by some suitable element $u \in \mathfrak{m}$, we have $ub \neq 0$ but $\mathfrak{m}ub = 0$. Let $a = ub$ and express $a = \sum_i s_i\omega_i$. There is at least one s_i, say s_1, that is nonzero. Now we can assume that S is reduced, so the localization $S[s_1^{-1}]$ is nonzero. Therefore, $\{a\} \cup \{\omega_i\}_{i=2}^{bd+d-n}$ is a basis of $N'[s_1^{-1}]$. After shrinking S, we can assume that N' admits a basis $\{\omega_i\}_{i=1}^{bd+d-n}$ such that $\mathfrak{m}\omega_1 = 0$.

Replace N' by N'/ω_1 and do the same trick. By induction, we see that after shrinking S sufficiently many times, there is a basis $\{\omega_i\}_{i=1}^{bd+d-n}$ of N' such that $\mathfrak{m}\omega_j \subseteq \operatorname{span}\{\omega_1, \dots, \omega_{j-1}\}$. Define a central series of unipotent group schemes

$$0 \lhd \mathscr{U}_{1,S} \lhd \mathscr{U}_{2,S} \lhd \cdots \lhd \mathscr{U}_{bd-n+r,S} = \operatorname{Fil}_1 \mathscr{G}_S \tag{6.38}$$

by setting the functors of points as

$$\mathscr{U}_{j,S} : T \in \mathbf{Alg}_S \mapsto \{g \in U[\mathfrak{m}F](T) : g(u_i) \in u_i + \operatorname{span}_S\{\omega_1, \dots, \omega_j\} \otimes_S T, 1 \le i \le d\}. \tag{6.39}$$

It follows easily from the construction that $\mathscr{U}_{j,S}/\mathscr{U}_{j,S} \simeq \mathbb{G}^d_{a,S}$. Therefore $\operatorname{Fil}_1 \mathscr{G}_S$ is S-split.

Replace $\operatorname{Quot}^r_{d,n}(R)$ by $\operatorname{Quot}^r_{d,n}(R) - \operatorname{Spec} S$ and repeat the above process. Since $\operatorname{Quot}^r_{d,n}(R)$ is finite type over k, this ends in finitely many steps, and yields the desired stratification. □

Lemma 6.10. *Notation as above. There is a Zariski stratification*

$$U \overset{c}{=} \bigsqcup_{\gamma\in\mathbf{C}} U_\gamma, \tag{6.40}$$

such that $\mathscr{G}_{U_\gamma}$ is a special U_γ-group scheme.

Proof. From Lemma 6.9 and Lemma 6.7(d), we see that there is a stratification $\{U_\gamma\}$ such that $\operatorname{Fil}_2 \mathscr{G}_{U_\gamma}$ is split and $\operatorname{gr}_3 \mathscr{G}_{U_\gamma}$ is isomorphic to $\mathrm{GL}_{d-r,U_\gamma}$. It is classical that $\mathrm{GL}_{d-r,U_\gamma}$ is special. Therefore $\mathscr{G}_{U_\gamma}$, as an extension of special group schemes, is special. □

Proof of Theorem 6.1. By Lemma 6.10, there is a stratification (6.40) on U such that the restriction of $\mathscr{G}_U$ to each stratum is special. It follows from Lemmas 6.8, 6.7, and the specialness of $\mathscr{G}_{U_\gamma}$ that

$$[U_\gamma][\mathrm{GL}[F]] = [\mathfrak{Q}_{U_\gamma}][\mathrm{GL}_{d-r}]\mathbb{L}^{bd^2-d(n-d)-(d-r)^2}. \tag{6.41}$$

Since various $\mathfrak{Q}_{U_\gamma}$ also form a Zariski stratification of $\mathfrak{Q}_U$, and U is a Zariski cover of $C^r_n(R)$, we have

$$[C^r_n(R)][\mathrm{GL}[F]] = [\mathfrak{Q}][\mathrm{GL}_{d-r}]\mathbb{L}^{bd^2-d(n-d)-(d-r)^2}. \tag{6.42}$$

Since $[\mathfrak{Q}] = [\operatorname{Quot}^r_{d,n}(R)][\mathrm{GL}_n]$, $[C^r_n(R)] = [\operatorname{Coh}^r_n(R)][\mathrm{GL}_n]$, and $[\mathrm{GL}[F]] = [\mathrm{GL}_d]\mathbb{L}^{bd^2}$ by Lemma 6.6, we obtain the formula (6.1). □

7. Functional equation

As our second major goal, we now prove Theorem 1.7. Assume $k = \mathbb{F}_q$ and $(*)$, and recall the notation in §4.1. This section is otherwise independent of §4 and uses different techniques.

Throughout the proof, we fix a finite-index subring $Z \subseteq R$ such that (i) Z is isomorphic to the power series ring $k[[X]]$ and (ii) K is a separable algebra over $Q := \operatorname{Frac} Z$. This is always possible: if $(T_i^{c_i})_{i=1}^s$ is the conductor of R, then we may take $Z = k[[\sum_{i=1}^s T_i^c]]$, where $c \geq \max_i c_i$ and c is not divisible by the characteristic of k. In this case, K is an s-fold product of finite separable field extensions of Q. We collect some basic facts, following [63, §2]. The ring extension $Q \subseteq K$ makes K a free module over Q of finite rank. We have a trace map $\operatorname{tr}_{K/Q} : K \to Q$ by defining $\operatorname{tr}_{K/Q}(\alpha)$ to be the trace of the multiplication-by-α map on K, viewed as a Q-linear map.

Let D be a generator for the different ideal of K/Q as an ideal of $\widetilde{R}$. Define the modified trace pairing $K \times K \to Q$ by $(x, y) \mapsto \operatorname{tr}_{K/Q}(D^{-1}xy)$; it is nondegenerate because K is a separable algebra over Q. For any fractional ideal I of R (namely, a rank-one R-lattice in K), define $I^\vee = \operatorname{Hom}_Z(I, Z)$ and view it as a fractional ideal of R by

$$I^\vee := \{x \in K : \operatorname{tr}_{K/Q}(D^{-1}xI) \subseteq Z\}. \tag{7.1}$$

The choice of D ensures that $(\widetilde{R})^\vee = \widetilde{R}$ as fractional ideals.

For any R-lattice L in a finite-dimensional K-vector space V, consider the $L^\vee := \operatorname{Hom}_Z(L, Z)$ with an R-module structure induced from the one on L. By the general tensor-hom duality, we have a canonical isomorphism of R-modules $L^\vee \simeq \operatorname{Hom}_R(L, R^\vee)$. We view $L^\vee$ as an R-lattice in $\operatorname{Hom}_K(V, K)$ using the natural inclusion $\operatorname{Hom}_R(L, R^\vee) \to \operatorname{Hom}_K(V, K)$. As a subset of $\operatorname{Hom}_K(V, K)$, we have $L^\vee = \{\theta \in \operatorname{Hom}_K(V, K) : \theta(L) \subseteq R^\vee\}$. We have the double dual property $(L^\vee)^\vee = L$ as lattices in K^d because any R-lattice is free of finite rank over Z, and $\operatorname{Hom}_Z(\cdot, Z)$ clearly has this property on free Z-modules of finite rank. For this reason, $R^\vee$ is a dualizing module of R, and we may take $\Omega = R^\vee$. For two R-lattices L_1, L_2 of V, we have $[L_1^\vee : L_2^\vee] = [L_2 : L_1]$.

We define a restricted lattice zeta function below. In this section, we drop the notation $|\cdot|_q$ throughout, i.e., every zeta function refers to point counts rather than the motives.

Definition 7.1. Let V be a K-vector space of dimension d, and fix R-lattices L, M in V. Define

$$Z_M(t; L) := \sum_{\substack{N \subseteq_R M \\ N \simeq_R L}} t^{[M:N]}, \text{ and } NZ_M(t; L) := (t; q)_d^s Z_M(t; L). \tag{7.2}$$

From the definition, $NZ_M(t; L)$ depends only on the isomorphism classes of L and M as R-modules. By [10, 44], $NZ_M(t; L)$ is a polynomial in t and the number of isomorphism classes of R-lattices in V is finite.

From now on, we define $V := K^d := K^{d\times 1}$, the space of column vectors, and identify $\operatorname{Hom}_K(V, K)$ with $K^{1\times d}$, the space of row vectors. For any fractional ideal I of R, we view $I^d := I^{d\times 1}$ as a lattice in V and $I^{1\times d}$ as a lattice in $K^{1\times d}$. We prove a refined version of Theorem 1.7. Recall $\delta = [\widetilde{R} : R]$.

Theorem 7.2. *Assuming the notation above, we have*

$$NZ_{\Omega^d}(t; L) = (q^{d^2} t^{2d})^\delta NZ_{\Omega^d}(q^{-d}t^{-1}; (L^\vee)^T) \tag{7.3}$$

for any R-lattice L in K^d, where $(L^\vee)^T$ denotes the image of $L^\vee$ under the transpose isomorphism $K^{1\times d} \to K^{d\times 1}$.

7.1. **Tools from harmonic analysis.** Let $V := K^d$ and fix R-lattices L, M in K^d throughout. We canonically identify $\operatorname{Hom}_R(L, M)$ with the set of $x \in \operatorname{Mat}_d(K)$ such that $xL \subseteq M$. We have a map $\operatorname{Hom}_R(L, M) \to \operatorname{Gr}_R(M)$ (the set of R-sublattices of M) by sending x to xL. Let $\operatorname{Inj}(L, M) = \operatorname{Hom}_R(L, M) \cap \operatorname{GL}_d(K)$. Then any sublattice $N \subseteq_R M$ such that $N \simeq_R L$ can be expressed as xL for some $x \in \operatorname{Inj}(L, M)$. Moreover, we know the overcount. Let $\operatorname{Aut}(L)$ be the subgroup of $\operatorname{GL}_d(K)$ consisting of x such that $xL = L$. For $x_1, x_2 \in \operatorname{Inj}(L, M)$, we have $x_1 L = x_2 L$ if and only if $x_2^{-1} x_1 \in \operatorname{Aut}(L)$.

We rewrite the definition of $Z_M(t;L)$ in terms of a Haar measure integral by using the substitution $N = xL$. Define $A := \mathrm{Mat}_d(K)$, so $A^\times = \mathrm{GL}_d(K)$. Let μ and dx denote the (additive) Haar measure on A with normalization $\mu(\mathrm{Mat}_d(\widetilde{R})) = 1$. Let $\mu^\times$ and $d^\times x$ denote the left-invariant (multiplicative) Haar measure on $A^\times$ with normalization $\mu^\times(\mathrm{GL}_d(\widetilde{R})) = 1/(q^{-1};q^{-1})_d^s$. For any $x \in A^\times$, define a multiplicative norm $\|\cdot\|_V$ by $\|x\|_V^{-1} := q^{[N:xN]}$ for any R-lattice N in V; this is independent of N. Define another multiplicative norm $\|\cdot\|_A$ on $A^\times$ by $\|x\|_A^{-1} := q^{[N:xN]} = q^{[N:Nx]}$ for any R-lattice N in A (viewed as K^{d^2}). We must be cautious about the distinction between $\|\cdot\|_A$ and $\|\cdot\|_V$. In fact, $\|x\|_A = \|x\|_V^d$. The measure $\mu^\times$ on $A^\times$ is bi-invariant and we have $d^\times x = \|x\|_A^{-1} dx$.

We use the change of variable $t = q^{-z}$ and define $\zeta_M(z;L) := Z_M(q^{-z};L)$ for complex numbers z with large enough real parts. For R-lattices N_1, N_2 of K^d, define $(N_1 : N_2) = q^{[N_1:N_2]}$. Then we have the following.

Lemma 7.3 ([10, Eq. (11)]). *We have*

$$\zeta_M(z;L) = \mu^\times(\mathrm{Aut}\, L)^{-1}(M:L)^{-z} \int_{x\in A^\times} \mathbf{1}_{\mathrm{Inj}(L,M)}(x)\|x\|_V^z d^\times x, \tag{7.4}$$

where $\mathbf{1}_{\mathrm{Inj}(L,M)}$ *is the indicator function of* $\mathrm{Inj}(L,M)$.

We recall the Fourier transform and a variant of a theorem of Tate. We note that $A = \prod_{i=1}^s \mathrm{Mat}_d(K_i)$, where $K_i := k((T_i))$. Since K_i/Q is a separable field extension, $\mathrm{Mat}_d(K_i)$ is a simple algebra over Q, and thus A is a semisimple algebra over Q. Let $\mathrm{tr}_{A/K} : A \to K$ be the usual trace map of $d \times d$ matrices over K, and define a modified trace $\mathrm{tr} : A \to Q$ by $\mathrm{tr}(x) := \mathrm{tr}_{K/Q}(D^{-1}\,\mathrm{tr}_{A/K}(x))$. Then the pairing on A (as a finite-dimensional Q-vector space) defined by $(x,y) \mapsto \mathrm{tr}(xy)$ is nondegenerate.

Fix a multiplicative character $\chi : Q = k((T)) \to \mathbb{C}^\times$ such that a fractional ideal I of Z (necessarily of the form $(T^n)k[[T]]$) lies in the kernel of χ if and only if $I \subseteq Z$ (i.e., $n \geq 0$). Concretely, if p is the characteristic of $k = \mathbb{F}_q$, we may define χ by

$$\chi\left(\sum_{n\in\mathbb{Z}} a_n T^n\right) = \exp\left(\frac{2\pi i}{p}\,\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_{-1})\right). \tag{7.5}$$

A Schwartz–Bruhat function on A is defined as a locally constant and compactly supported function from A to $\mathbb{C}$. Given a Schwartz–Bruhat function Φ on A, its Fourier transform is defined as

$$\widehat{\Phi}(y) = \int_A \Phi(x)\chi(\mathrm{tr}(xy))dx, \tag{7.6}$$

which is again a Schwartz–Bruhat function on A. We remark that in general, the Fourier transform is defined on any finite-dimensional vector space over Q and depends on the nondegenerate pairing, the character χ and the (normalization of) the additive Haar measure dx.

Define the zeta integral associated to a Schwartz–Bruhat function Φ on A by

$$Z(\Phi;z) := \int_{x\in A^\times} \Phi(x)\|x\|_A^z d^\times x. \tag{7.7}$$

The following theorem is a generalization of a theorem in Tate's thesis and is crucial in our proof of Theorem 1.7.

Theorem 7.4 ([10, Prop. 2]). *Assume the setting above. Given any Schwartz–Bruhat functions* Φ, Ψ *on* A*, the zeta integrals* $Z(\Phi;z)$ *and* $Z(\Psi;z)$ *admit meromorphic continuations to all of* $z \in \mathbb{C}$*, and satisfy*

$$\frac{Z(\Phi;z)}{Z(\Psi;z)} = \frac{Z(\widehat{\Phi};1-z)}{Z(\widehat{\Psi};1-z)}. \tag{7.8}$$

7.2. **Fourier transform of an indicator function.** Using Lemma 7.3, the definition of the zeta integral, the definition $\mathrm{Inj}(L, M) = \mathrm{Hom}_R(L, M) \cap A^\times$, and the relation $\|x\|_A = \|x\|_V^d$, we have

$$\zeta_M(z; L) = \mu^\times(\mathrm{Aut}\, L)^{-1}(M : L)^{-z} Z(\mathbf{1}_{\mathrm{Hom}_R(L,M)}; z/d). \tag{7.9}$$

To apply Theorem 7.4 to obtain a functional equation for $\zeta_M(z; L)$, we need to compute the Fourier transform of $\mathbf{1}_{\mathrm{Hom}(L,M)}$ when $M = \Omega^{d\times 1}$. From now on, we denote by μ the Haar measure on K normalized by $\mu(\widetilde{R}) = 1$; by abuse of notation, we let μ also denote the product measure it induces on $K^{d\times 1}, K^{1\times d}$ or $A = K^{d\times d}$, which is consistent with our notation for the additive Haar measure on A.

Lemma 7.5. *Let L be an R-lattice in V, and $M = \Omega^{d\times 1}$. Then*

$$\widehat{\mathbf{1}}_{\mathrm{Hom}_R(L,M)} = \mu(L^\vee)^d\, \widehat{\mathbf{1}}_{L^{1\times d}}. \tag{7.10}$$

Proof. First, note that $\mathrm{Hom}_R(L, M) = \mathrm{Hom}_R(L, \Omega)^{d\times 1} = (L^\vee)^{d\times 1}$; this means $\mathrm{Hom}_R(L, M)$ is an R-submodule of $A = K^{d\times d}$ consisting of matrices whose rows lie in $L^\vee$.

Now fix $y \in A$, and denote by $y_j \in K^{d\times 1}$ the j-th column of y. By letting x_i denote the i-th row of $x \in \mathrm{Hom}_R(L, M)$ and using the multiplicativity of χ, we have

$$\widehat{\mathbf{1}}_{\mathrm{Hom}_R(L,M)}(y) = \int_{x\in \mathrm{Hom}_R(L,M)} \chi(\mathrm{tr}(xy))dx \tag{7.11}$$

$$= \int_{x_1,\dots,x_d\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}\,\mathrm{tr}_{A/K}((x_i y_j)_{ij})))dx_1 \dots dx_d \tag{7.12}$$

$$= \int_{x_1,\dots,x_d\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}(x_1 y_1 + \cdots + x_d y_d)))dx_1 \dots dx_d \tag{7.13}$$

$$= \prod_{i=1}^{d} \int_{x_i\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i))dx_i. \tag{7.14}$$

It suffices to show that $\int_{x_i\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i))dx_i = \mu(L^\vee)\mathbf{1}_L(y_i)$. To prove the claim, we recall from the definition of dual lattices and the double dual property that

$$L = \{v \in K^{d\times 1} : uv \in R^\vee \text{ for } u \in L^\vee\}. \tag{7.15}$$

We also recall that as a subset of K, we have $R^\vee = \{\alpha : \mathrm{tr}_{K/Q}(D^{-1}\alpha R) \subseteq Z\}$. Hence, any $\alpha \in R^\vee$ satisfies $\mathrm{tr}_{K/Q}(D^{-1}\alpha) \in Z$. As a result, if $y_i \in L$, then for any $x_i \in L^\vee$, we have $x_i y_i \in R^\vee$, so that $\mathrm{tr}_{K/Q}(D^{-1}x_i y_i) \in Z$. Since χ is trivial on Z, we have $\int_{x_i\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i))dx_i = \int_{x_i\in L^\vee} 1\, dx_i = \mu(L^\vee)$ if $y_i \in L$.

It remains to show that $\int_{x_i\in L^\vee} \chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i))dx_i = 0$ if $y_i \notin L$. We claim that

$$x_i \mapsto \chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i)) \tag{7.16}$$

is a nontrivial character on $L^\vee$; if this claim is proved, then the proof is complete because the integral of a nontrivial character is necessarily zero. We proceed by contradiction. Assume that $\chi(\mathrm{tr}_{K/Q}(D^{-1}x_i y_i))$ is a trivial character on $x_i \in L^\vee$. Then the fractional ideal of Z in Q given by $\mathrm{tr}_{K/Q}(D^{-1}L^\vee y_i)$ is in the kernel of χ. From our construction of χ, we must have $\mathrm{tr}_{K/Q}(D^{-1}L^\vee y_i) \subseteq Z$. Since $L^\vee y_i$ is a fractional ideal of R in K, any $\alpha \in L^\vee y_i$ satisfies $D^{-1}\alpha R \subseteq D^{-1}L^\vee y_i$, so that $\mathrm{tr}_{K/Q}(D^{-1}\alpha R) \subseteq Z$. By the characterization of $R^\vee$ as a subset of K, we have $\alpha \in R^\vee$. It follows that $L^\vee y_i \in R^\vee$. By (7.15), we have $y_i \in L$, a contradiction. □

Before we prove Theorem 7.2, we need one more observation.

Lemma 7.6. *For any R-lattice L in $K^{d\times 1}$, we have $\mu^\times(\mathrm{Aut}\, L) = \mu^\times(\mathrm{Aut}((L^\vee)^T))$.*

Proof. We claim that $\mathrm{Aut}((L^\vee)^T) = (\mathrm{Aut}\, L)^T$. If the claim is proved, since the transpose isomorphism $A^\times \to A^\times$ preserves the measure $\mu^\times$, the lemma follows.

Noting that $\mathrm{Hom}_R(L,L)$ is a subalgebra of A and $\mathrm{Aut}\,L = \mathrm{Hom}_R(L,L)^\times$, it suffices to prove that $\mathrm{Hom}_R((L^\vee)^T,(L^\vee)^T) = \mathrm{Hom}_R(L,L)^T$. Equivalently, we claim $x \in \mathrm{Mat}_d(K)$ satisfies $xL \subseteq L$ if and only if $x^T(L^\vee)^T \subseteq (L^\vee)^T$, which is itself equivalent to $L^\vee x \subseteq L^\vee$.

We recall from the definition of $L^\vee$ and the double dual property that

$$L^\vee = \{u \in K^{1\times d} : uL \subseteq R^\vee\}, \quad L = \{v \in K^{d\times 1} : L^\vee v \subseteq R^\vee\}. \tag{7.17}$$

It follows from the second equality that $xL \subseteq L$ if and only if $L^\vee xL \subseteq R^\vee$, and it follows from the first equality that $L^\vee x \subseteq L^\vee$ if and only if $L^\vee xL \subseteq R^\vee$. This proves the claim. □

7.3. **Proofs of Theorem 7.2 and Theorem 1.7.** We now prove Theorem 7.2 and Theorem 1.7.

Proof of Theorem 7.2. Let $M = (R^\vee)^{d\times 1}$ and fix an R-lattice L in $V = K^{d\times 1}$. We remark that even though the zeta function $Z_M(t;L)$ depends only on the isomorphism classes of M, L as R-modules, we need to fix the embeddings $M, L \subseteq V$ throughout our proof. Consider the equation (7.8) with $\Phi = \mathbf{1}_{\mathrm{Hom}_R(L,M)}$ and $\Psi = \mathbf{1}_{\mathrm{Mat}_d(\widetilde{R})}$, and substitute z with z/d. The strategy of the proof is to express both sides of the equation above in terms of the restricted lattice zeta function using (7.9) and (7.10).

We start with the left-hand side. By (7.9), we have

$$Z(\Phi; z/d) = \mu^\times(\mathrm{Aut}\,L)(M:L)^z \zeta_M(z;L). \tag{7.18}$$

Note that the normalization of every measure and the Fourier transform involved in (7.9) depends on $K, Q, \widetilde{R}$ but not on R. In particular, we may apply (7.9) with $R = \widetilde{R}$ and $L = M = \widetilde{R}^d$. It follows that

$$Z(\Psi; z/d) = \mu^\times(\mathrm{GL}_d(\widetilde{R}))\zeta^{\widetilde{R}}_{\widetilde{R}^d}(z;\widetilde{R}^d), \tag{7.19}$$

where the notation $\zeta^{\widetilde{R}}_{\widetilde{R}^d}$ indicates that the lattice zeta function is with respect to the ground ring $\widetilde{R}$. Since every $\widetilde{R}$-lattice is isomorphic to $\widetilde{R}^d$, the function $\zeta^{\widetilde{R}}_{\widetilde{R}^d}(z;\widetilde{R}^d)$ is simply $\zeta^{\widetilde{R}}_{\widetilde{R}^d}(z)$, which by Solomon's formula equals to $1/(q^{-z};q)^s_d$. Substituted into the above equation, it follows from the definition of $NZ_M(t;L)$ that

$$\frac{Z(\Phi; z/d)}{Z(\Psi; z/d)} = \frac{\mu^\times(\mathrm{Aut}\,L)}{\mu^\times(\mathrm{GL}_d(\widetilde{R}))}(M:L)^z NZ_M(q^{-z};L). \tag{7.20}$$

We now consider the right-hand side of (7.8) with $z \mapsto z/d$. By (7.10), we have

$$Z(\widehat{\Phi}; 1-z/d) = \mu(L^\vee)^d Z(\mathbf{1}_{L^{1\times d}}; 1-z/d). \tag{7.21}$$

By the definition of the zeta integral and the fact that matrix transposition $x \mapsto x^T$ preserves the norm $\|\cdot\|_A$ and the measure $d^\times x$, we have

$$Z(\widehat{\Phi}; 1-z/d) = \mu(L^\vee)^d Z(\mathbf{1}_{(L^T)^{d\times 1}}; 1-z/d). \tag{7.22}$$

By the double dual property, we have $(L^T)^{d\times 1} = \mathrm{Hom}_R((L^\vee)^T, M)$. It then follows from (7.9) that

$$Z(\widehat{\Phi}; 1-z/d) = \mu(L^\vee)^d \mu^\times(\mathrm{Aut}((L^\vee)^T))(M:(L^\vee)^T)^{d-z}\zeta_M(d-z;(L^\vee)^T). \tag{7.23}$$

To compute $Z(\widehat{\Psi}; 1-z/d)$, we again consider Ψ as a special case of Φ with $R = \widetilde{R}$ and $L = M = \widetilde{R}^d$. Substituting this into the above equation and using the self-dualness of $\widetilde{R}$, the normalization $\mu(\widetilde{R}) = 1$ and Solomons' formula, we have

$$Z(\widehat{\Psi}; 1-z/d) = \mu^\times(\mathrm{GL}_d(\widetilde{R}))(q^{-(d-z)};q)^{-s}_d. \tag{7.24}$$

Combining the above equations and using the definition of $NZ_M(t;(L^\vee)^T)$, we have

$$\frac{Z(\widehat{\Phi}; 1-z/d)}{Z(\widehat{\Psi}; 1-z/d)} = \frac{\mu^\times(\mathrm{Aut}((L^\vee)^T))}{\mu^\times(\mathrm{GL}_d(\widetilde{R}))}\mu(L^\vee)^d(M:(L^\vee)^T)^{d-z}NZ_M(q^{-(d-z)};(L^\vee)^T). \tag{7.25}$$

By Theorem 7.4, the formulas (7.20) and (7.25) are equal. Equating them, applying Lemma 7.6, removing the common factor $\mu^\times(\mathrm{GL}_d(\widetilde{R}))$, simplifying using the general property $(L_1 : L_2) = \mu(L_1)/\mu(L_2)$, and noting that $\mu(L^\vee) = \mu((L^\vee)^T)$, we get

$$\frac{NZ_M(q^{-z}; L)}{NZ_M(q^{-(d-z)}; (L^\vee)^T)} = \mu(L)^z \mu(L^\vee)^z \mu(M)^{d-2z}. \tag{7.26}$$

As the final step, we simplify the right-hand side of (7.26) using properties of duals. Recall the general property $(L_1^\vee : L_2^\vee) = (L_2 : L_1)$. Substituting $L_1 = L, L_2 = (\widetilde{R})^{d\times 1}$ and using the self-dualness $(\widetilde{R}^{d\times 1})^\vee = \widetilde{R}^{1\times d}$, the general property $(L_1 : L_2) = \mu(L_1)/\mu(L_2)$ and the normalization $\mu(\widetilde{R}) = 1$, we have $\mu(L)\mu(L^\vee) = 1$. Using $M = (R^\vee)^{d\times 1}$, we similarly have

$$\mu(M) = \mu(R^\vee)^d = (R^\vee : \widetilde{R})^d = (\widetilde{R} : R)^d = q^{\delta d}, \tag{7.27}$$

where we recall that $\delta = [\widetilde{R} : R]$ is the Serre invariant. We thus get

$$\frac{NZ_M(q^{-z}; L)}{NZ_M(q^{-(d-z)}; (L^\vee)^T)} = q^{(d^2-2dz)\delta}. \tag{7.28}$$

Setting $t = q^{-z}$ completes the proof of Theorem 7.2. □

Proof of Theorem 1.7. Let S denote the set of isomorphism classes of R-lattices in V. Note that $NZ_M(t; [L])$ is well-defined for $[L] \in S$. By the double dual property, $[L] \mapsto [(L^\vee)^T]$ gives an involution on S. Theorem 1.7 then follows from Theorem 7.2 and summing over all $[L] \in S$. □

Remark 7.7. One might be able to obtain a motivic analogue of Theorem 7.2 using a motivic version of the Fourier method, e.g., [13]. But even so, to attack Conjecture 1.6, it needs to be done relatively with the base being the stack of rank d torsion-free modules over R.

8. The $y^2 = x^n$ singularity

In this section, we apply Proposition 4.7 to prove the point-counting versions of Theorems 1.8 and 1.10. We first make an alternative formulation of (4.12) that is useful in our cases.

8.1. **Boundary R-lattices.** Define the set of **boundary R-lattices** for $M/\underset{\sim}{M}$ to be

$$\partial_{\underset{\sim}{M}}(M) := \{\underset{\sim}{M} \subseteq L \subseteq_R M : \widetilde{R}L \cap M = L\}. \tag{8.1}$$

We note that lattices in $\partial_{\underset{\sim}{M}}(M)$ necessarily lie in M, unlike the lattices in $\widetilde{\partial}_{\underset{\sim}{M}}(M)$; this turns out to be a convenient feature later. Nevertheless, $\partial_{\underset{\sim}{M}}(M)$ is in a canonical bijection with $\widetilde{\partial}_{\underset{\sim}{M}}(M)$.

Lemma 8.1. *The map $L \mapsto \widetilde{R}L$ gives a bijection $\partial_{\underset{\sim}{M}}(M) \to \widetilde{\partial}_{\underset{\sim}{M}}(M)$ with inverse $\widetilde{L} \mapsto \widetilde{L} \cap M$.*

Proof. If $L \in \partial_{\underset{\sim}{M}}(M)$, then $\widetilde{R}L \in \widetilde{\partial}_{\underset{\sim}{M}}(M)$ because $L \in E_R(\widetilde{R}L; M)$. Therefore, $L \mapsto \widetilde{R}L$ is well-defined. Conversely, if $\widetilde{L} \in \widetilde{\partial}_{\underset{\sim}{M}}(M)$, assume $N \in E_R(\widetilde{L}; M)$. Then we have $N \subseteq \widetilde{L} \cap M =: L$ and $\widetilde{R}N = \widetilde{L}$. It follows that $\widetilde{R}L = \widetilde{L}$, so that $L \in \partial_{\underset{\sim}{M}}(M)$. As a result, $\widetilde{L} \mapsto \widetilde{L} \cap M$ is also well-defined.

It is clear that both maps are inverse to each other, finishing the proof. □

The alternative recipe to parametrize R-sublattices of M is as follows:

- Choose $L_b \in \partial_{\underset{\sim}{M}}(M)$;
- Choose $\widetilde{N}$ using the data at the right-hand side of (4.5) with $\widetilde{L} := \widetilde{R}L_b$, and independently choose $L \in E_R(\widetilde{R}L_b; L_b)$.

When $k = \mathbb{F}_q$, we get the following version of (4.12):

$$|NZ_M^R(t)|_q = \sum_{L_b \in \partial_{\underset{\sim}{M}}(M)} \left(\prod_{i=1}^{s} (t;q)_{\mathrm{rk}_i(\widetilde{R}L_b/\underset{\sim}{M})}\right) \sum_{L \in E_R(\widetilde{R}L_b; L_b)} t^{[L_b:L]}. \tag{8.2}$$

8.2. **Structure of $R^{(2,n)}$.** Fix $m \geq 1$ and a field k. Recall $R^{(2,2m+1)} := k[[X,Y]]/(Y^2 - X^{2m+1})$ and $R^{(2,2m)} := k[[X,Y]]/(Y(Y-X^m))$. We first review these rings in terms of the notation in §4.1.

- For $R = R^{(2,2m+1)}$, we have $s = 1$, and we identify the normalization map with
$$R = k[[T^2, T^{2m+1}]] \hookrightarrow k[[T]] = \widetilde{R}. \tag{8.3}$$
The conductor is $\mathfrak{c} = (T^{2m})\widetilde{R}$. We have $R/\mathfrak{c} \simeq k[[X]]/X^m$, where $X = T^2$.
- For $R = R^{(2,2m)}$, we have $s = 2$, and we identify the normalization map with
$$R = k[[T_1^m, T_2^m, T_1 + T_2]] \hookrightarrow k[[T_1]] \times k[[T_2]] = \widetilde{R}. \tag{8.4}$$
The conductor is $\mathfrak{c} = (T_1^{2m}, T_2^{2m})\widetilde{R}$. We have $R/\mathfrak{c} \simeq k[[X]]/X^m$, where $X = T (:= T_1 + T_2)$.

We crucially note that in both cases, $R/\mathfrak{c}$ is a DVR quotient.

8.3. **Explicit parametrization of extension fibers.** We now explicitly parametrize $E_R(\widetilde{L})$ for an arbitrary $\widetilde{R}$-lattice $\widetilde{L}$ if $R = R^{(2,2m+1)}$ or $R^{(2,2m)}$, which serves two purposes. First, it will be needed to compute $|Z^R_{\widetilde{R}^d}(t)|_q$, the "easier" part of the point-counting versions of Theorems 1.8 and 1.14. Second, we will use it to parametrize $E_R(\widetilde{L}; M)$, which will be needed to compute $|Z_{R^d}(t)|_q$, the "harder" part.

Our general strategy is to reduce the above problem to a special case of Lemma 3.2 with $A = R/\mathfrak{c}$ and V_2 free (so the set $\mathrm{Hom}_A(V_2, V_1/W')$ is easy to describe). To achieve it, we prove Lemmas 8.2 and 8.3 below. In the proofs, it is useful to note that
$$\widetilde{R} = R + TR, \text{ if } R = R^{(2,2m+1)} \tag{8.5}$$
and
$$\widetilde{R} = R + e_1 R = R + e_2 R = e_1 R + e_2 R, \text{ if } R = R^{(2,2m)}. \tag{8.6}$$
As a result, analogous equations hold if $\widetilde{R}$ is replaced by $\widetilde{R}L$ and R is replaced by L.

Lemma 8.2. *Let $R = R^{(2,2m)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Then an R-lattice $L \subseteq \widetilde{L}$ satisfies $\widetilde{R}L = \widetilde{L}$ if and only if $L + e_i\widetilde{L} = \widetilde{L}$ for $i = 1, 2$.*

Proof. We note that the following decompositions are actually direct sums:
$$\widetilde{R}L = e_1 L \oplus e_2 L, \tag{8.7}$$
$$\widetilde{L} = e_1 \widetilde{L} \oplus e_2 \widetilde{L}. \tag{8.8}$$
Hence, $\widetilde{R}L = \widetilde{L}$ if and only if $e_i L = e_i \widetilde{L}$ for $i = 1, 2$. Since multiplication by e_i is precisely the projection map from $\widetilde{L}$ to $e_i\widetilde{L}$ with respect to the decomposition (8.8), the condition $e_i L = e_i \widetilde{L}$ is equivalent to $L + e_j \widetilde{L} = \widetilde{L}$, where $(i,j) = (1,2)$ or $(2,1)$. □

Lemma 8.3. *Let $R = R^{(2,2m+1)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Fix an R-lattice $F_{\widetilde{L}} \subseteq \widetilde{L}$ such that $\widetilde{R}F_{\widetilde{L}} = \widetilde{L}$. Then an R-lattice $L \subseteq \widetilde{L}$ satisfies $\widetilde{R}L = \widetilde{L}$ if and only if $L + T \cdot F_{\widetilde{L}} = \widetilde{L}$.*

Proof. Assume $L + TF_{\widetilde{L}} = \widetilde{L}$. To prove $L + TL = \widetilde{L}$, since $T^n F_{\widetilde{L}} \subseteq L$ for $n \gg 0$, it suffices to show that $\widetilde{L} = L + TL + T^n F_{\widetilde{L}}$ for all $n \geq 1$. We prove it by induction on n. The base case $n = 1$ follows from the assumption. For the induction step, assume $n \geq 1$ and $\widetilde{L} = L + TL + T^n F_{\widetilde{L}}$, then we have
$$\widetilde{L} = L + TL + T^n F_{\widetilde{L}} \subseteq L + TL + T^n \widetilde{L} = L + TL + T^n(L + TF_{\widetilde{L}}) = L + TL + T^{n+1}F_{\widetilde{L}}, \tag{8.9}$$
where the last equality follows from that $T^n L \subseteq L$ if n is even and $T^n L \subseteq TL$ if n is odd. (To see this, recall that $T^2 \in R$ and L is an R-lattice.)

Conversely, assume $L + TL = \widetilde{L}$. Then we have
$$\widetilde{L} = L + TL \subseteq L + T\widetilde{L} = L + T(F_{\widetilde{L}} + TF_{\widetilde{L}}) = L + TF_{\widetilde{L}} + T^2 F_{\widetilde{L}}. \tag{8.10}$$

Again, to prove $\widetilde{L} = L + TF_{\widetilde{L}}$, it suffices to prove that $\widetilde{L} = L + TF_{\widetilde{L}} + T^{2^n}F_{\widetilde{L}}$ for $n \geq 1$. We have just proved the base case $n = 1$. To proceed with induction, assume $n \geq 1$ and $\widetilde{L} = L + TF_{\widetilde{L}} + T^{2^n}F_{\widetilde{L}}$, then we have

$$\begin{aligned}\widetilde{L} &= L + TF_{\widetilde{L}} + T^{2^n}F_{\widetilde{L}} \subseteq L + TF_{\widetilde{L}} + T^{2^n}\widetilde{L}\\ &= L + TF_{\widetilde{L}} + T^{2^n}(L + TF_{\widetilde{L}} + T^{2^n}F_{\widetilde{L}}) = (L + T^{2^n}L) + (TF_{\widetilde{L}} + T^{2^n+1}F_{\widetilde{L}}) + T^{2^{n+1}}F_{\widetilde{L}}\\ &= L + TF_{\widetilde{L}} + T^{2^{n+1}}F_{\widetilde{L}}.\end{aligned} \tag{8.11}$$

□

From now on, if $R = R^{(2,2m+1)}$, we fix a *free* R-sublattice $F_{\widetilde{L}}$ of $\widetilde{L}$ such that $\widetilde{R}F_{\widetilde{L}} = \widetilde{L}$. For instance, one may let $F_{\widetilde{L}}$ be the R-lattice generated by $u_1, \dots, u_d$, where $\{u_1, \dots, u_d\}$ is an $\widetilde{R}$-basis of $\widetilde{L}$. As is foreshadowed in Lemma 8.3, our parametrization of $E_R(\widetilde{L})$ will depend on the choice of $F_{\widetilde{L}}$.

We introduce some notation needed in our parametrization for $E_R(\widetilde{L})$.

Notation 8.4. Define $A := R/\mathfrak{c}$, $\widetilde{A} := \widetilde{R}/\mathfrak{c}$ and recall that $A \simeq k[[X]]/X^m$. Consider the $\widetilde{A}$-module $V := \widetilde{L}/\mathfrak{c}\widetilde{L}$. We define the following A-submodules of V once $F_{\widetilde{L}}$ is fixed:

- If $R = R^{(2,2m+1)}$, let $V_2 = F_{\widetilde{L}}/\mathfrak{c}\widetilde{L}$ and $V_1 = TV_2$.
- If $R = R^{(2,2m)}$, let $V_i = e_iV$ for $i = 1, 2$.

We note that V_i are free A-modules of rank d, and $V = V_1 \oplus V_2$. For the direct sum assertion, it suffices to observe for $R = R^{(2,2m+1)}$ that $\widetilde{R}/\mathfrak{c}$ is a free module of rank 2 over $R/\mathfrak{c}$ with basis $\{1, T\}$.

By Lemma 4.1, we may identify elements of $E_R(\widetilde{L})$ with A-submodules of V using

$$L \mapsto W_L := L/\mathfrak{c}\widetilde{L}, \text{ for } L \text{ with } \mathfrak{c}\widetilde{L} \subseteq L \subseteq \widetilde{L}. \tag{8.12}$$

We always have $[\widetilde{L} : L] = [V : W_L]$.

We now state our parametrization.

Lemma 8.5. *Let $R = R^{(2,2m+1)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Fix $F_{\widetilde{L}}$ as above. Then in the notation above, there is a bijection*

$$\begin{aligned}E_R(\widetilde{L}) &\to \{(W', \varphi) : W' \subseteq_A V_1, \varphi \in \operatorname{Hom}_A(V_2, V_1/W')\},\\ L &\mapsto (W'_L, \varphi_L),\end{aligned} \tag{8.13}$$

where $W'_L := W_L \cap V_1$ and φ_L is the composition $V_2 \to V \to V/W_L \simeq V_1/W'_L$. Moreover, we have $[\widetilde{L} : L] = [V_1 : W']$.

Proof. Taking Lemma 8.3 modulo $\mathfrak{c}\widetilde{L}$, we conclude that an R-lattice L with $\mathfrak{c}\widetilde{L} \subseteq L \subseteq \mathfrak{c}L$ is in $E_R(\widetilde{L})$ if and only if $W_L + V_1 = V$. Then the result follows from Lemma 3.2. □

Lemma 8.6. *Let $R = R^{(2,2m)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Then in the notation above, there is a bijection*

$$\begin{aligned}E_R(\widetilde{L}) &\to \{(W', \varphi) : W' \subseteq_A V_1, \varphi \in \operatorname{Surj}_A(V_2, V_1/W')\},\\ L &\mapsto (W'_L, \varphi_L),\end{aligned} \tag{8.14}$$

where $W'_L := W_L \cap V_1$ and φ_L is the composition $V_2 \to V \to V/W_L \simeq V_1/W'_L$. Moreover, we have $[\widetilde{L} : L] = [V_1 : W']$.

Proof. Taking Lemma 8.2 modulo $\mathfrak{c}\widetilde{L}$, we conclude that an R-lattice L with $\mathfrak{c}\widetilde{L} \subseteq L \subseteq \mathfrak{c}L$ is in $E_R(\widetilde{L})$ if and only if $W_L + V_1 = W_L + V_2 = V$. We first classify W_L with $W_L + V_1 = V$ using Lemma 3.2. Under this classification, note that φ_L is the natural map $V_2 \to V/W_L$, so that $W_L + V_2 = V$ if and only if φ_L is surjective. □

Remark 8.7. A finitely generated A-module is nothing but a $k[[X]]$-module whose type λ satisfies $\lambda_1 \leq m$. In later discussions, we often identify the classifying set on the right-hand side of (8.13) with

$$\{(W', \varphi) : W' \subseteq_{k[[X]]} V_1, \varphi \in \operatorname{Hom}_{k[[X]]}(k[[X]]^d, V_1/W')\}, \tag{8.15}$$

and similarly for (8.14).

Our parametrizations above immediately imply the "easier" part of the point-counting versions of Theorems 1.8 and 1.10. Recall §3.2.

Proposition 8.8. *Let $k = \mathbb{F}_q$ and $R = R^{(2,2m+1)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Assume $k = \mathbb{F}_q$. Then*

$$|NZ^R_{\widetilde{R}^d}(t)|_q = \sum_{\mu \subseteq (m^d)} g_\mu^{(m^d)}(q)\,(q^d t)^{|\mu|}. \tag{8.16}$$

Proof. We note that $V_1 = A^d$ is a $k[[X]]$-module of type (m^d). In Lemma 8.5, letting μ be the cotype of W' in V_1, we get

$$\sum_{L \in E_R(\widetilde{L})} t^{[\widetilde{L}:L]} = \sum_{\mu \subseteq (m^d)} g_\mu^{(m^d)}(q)\, q^{d|\mu|} t^{|\mu|}. \tag{8.17}$$

The desired formula then follows from Remark 4.9. □

Proposition 8.9. *Let $R = R^{(2,2m)}$, and $\widetilde{L}$ be an $\widetilde{R}$-lattice. Assume $k = \mathbb{F}_q$. Then*

$$|NZ^R_{\widetilde{R}^d}(t)|_q = \sum_{\mu \subseteq (m^d)} g_\mu^{(m^d)}(q)\,(q^d t)^{|\mu|} \frac{(q^{-1};q^{-1})_d}{(q^{-1};q^{-1})_{d-\mu_1'}}. \tag{8.18}$$

Proof. The proof is analogous to the proof of Proposition 8.8, except that we need to multiply by the probability that φ be surjective, which is $(q^{-1};q^{-1})_d/(q^{-1};q^{-1})_{d-\mu_1'}$ by Lemma 3.6. □

We now describe $E_R(\widetilde{L}; M)$ as a subset of $E_R(\widetilde{L})$ using the parametrization data in Lemmas 8.5 or 8.6. Assume $E_R(\widetilde{L}; M)$ is nonempty, so that $L_b := \widetilde{L} \cap M$ is in $E_R(\widetilde{L})$, and $E_R(\widetilde{L}; M) = E_R(\widetilde{L}; L_b)$. The following lemma describes $E_R(\widetilde{L}; L_b)$ for $L_b \in E_R(\widetilde{L})$.

Lemma 8.10. *Let $R = R^{(2,n)}$, and let $\widetilde{L}$ be an $\widetilde{R}$-lattice. Suppose $L_b \in E_R(\widetilde{L})$ is classified by $(W'_{L_b}, \varphi_{L_b})$ under Lemma 8.5 or Lemma 8.6. Then $E_R(\widetilde{L}; L_b)$ consists of lattices $L \in E_R(\widetilde{L})$ whose classifying datum (W'_L, φ_L) satisfies*

- *$W'_L \subseteq W'_{L_b}$;*
- *φ_L is a lift of φ_{L_b}, i.e., φ_{L_b} is the composition $V_2 \xrightarrow{\varphi_L} V_1/W'_L \to V_1/W'_{L_b}$.*

Proof. For $L \in E_R(\widetilde{L})$, we have $L \in E_R(\widetilde{L}; L_b)$ if and only if $W_L \subseteq W_{L_b}$. This is equivalent to (a)(b) by the definition of both sides of the bijection $W_L \leftrightarrow (W'_L, \varphi_L)$ in Lemma 3.2 and its proof. □

Remark 8.11. Assume $k = \mathbb{F}_q$. Given $W'_L \subseteq W'_{L_b}$ and φ_{L_b}, the number of $\varphi_L \in \operatorname{Hom}_A(V_2, V_1/W'_L)$ that lifts φ_{L_b} is clearly $q^{d[W'_{L_b}:W'_L]}$. However, if $R = R^{(2,2m)}$, we need to count the number of surjective lifts as per Lemma 8.6. Fortunately, by Lemma 3.1, every pair of surjections φ_{L_b}'s are related by some element in $\mathrm{GL}(V_2) = \mathrm{GL}_d(A)$, so the number of surjective lifts of φ_{L_b} is constant as φ_{L_b} varies. Therefore, we can compute this number by

$$\frac{\#\operatorname{Surj}_A(V_2, V_1/W'_L)}{\#\operatorname{Surj}_A(V_2, V_1/W'_{L_b})}, \tag{8.19}$$

for which we can invoke Lemma 3.6.

8.4. **Sublattices of R^d.** Define $F := R^d$. We now apply (8.2) with $M = F$ and $\underset{\sim}{M} = \mathfrak{c}F$ to parametrize R-sublattices of F. It remains to describe the elements $L_b \in \partial_{\mathfrak{c}F}(F)$, and determine $\mathrm{rk}_i(\widetilde{R}L_b/\mathfrak{c}F)$ and $E_R(\widetilde{R}L_b; L_b)$ in terms of L_b. The following lemma describes $\partial_{\mathfrak{c}F}(F)$.

Lemma 8.12. *For $R = R^{(2,2m+1)}$ or $R^{(2,2m)}$, we have $\partial_{\mathfrak{c}F}(F) = \{\mathfrak{c}F \subseteq L \subseteq_R F\}$.*

Proof. By the definition in (8.1), it suffices to prove that every $\mathfrak{c}F \subseteq L \subseteq_R F$ satisfies $\widetilde{R}L \cap F = L$. We recall the notation $A, \widetilde{A}, V, W_{(\cdot)}$ from Notation 8.4 (with the ambient lattice $\widetilde{L}$ replaced by $\widetilde{F}$) and work modulo $\mathfrak{c}F$. Then it is enough to show that as A-submodules of V, we have $\widetilde{A}W \cap W_F = W$ for any $W \subseteq_A W_F$. But this is true because $W_F = A^d$ and $V = \tilde{A}^d$ canonically, and A is a direct summand of $\widetilde{A}$ as an A-module. Indeed, $\widetilde{A} = A \oplus TA$ if $R = R^{(2,2m+1)}$, and $\widetilde{A} = A \oplus T_1 A = A \oplus T_2 A$ if $R = R^{(2,2m)}$. ☐

As a result, we have a bijection $\partial_{\mathfrak{c}F}(F) \to \{W \subseteq_{k[[X]]} W_F\}$ defined by $L \mapsto W_L := L/\mathfrak{c}F$. Note that $W_F \simeq_A A^d$ is of type (m^d). From now on, fix a boundary lattice $L_b \in \partial_{\mathfrak{c}F}(F)$, and denote $W_b := L_b/\mathfrak{c}F$. Let $\lambda = \lambda_{k[[X]]}(F/L_b) = \lambda_{k[[X]]}(W_F/W_b)$ be the cotype of W_b in W_F. The type of W_b is then given by $(m^d) - \lambda$. We now describe $\mathrm{rk}_i(\widetilde{R}L_b/\mathfrak{c}F)$ and $E_R(\widetilde{R}L_b; L_b)$ solely in terms of the partition $\lambda \subseteq (m^d)$.

Lemma 8.13. *Assume the notation above. If $R = R^{(2,2m+1)}$, then $\mathrm{rk}(\widetilde{R}L_b/\mathfrak{c}F) = d - \lambda'_m$. If $R = R^{(2,2m)}$, then $\mathrm{rk}_i(\widetilde{R}L_b/\mathfrak{c}F) = d - \lambda'_m$ for $i = 1, 2$.*

Proof. We note that $\widetilde{R}L_b/\mathfrak{c}F = \widetilde{A}W_b$ and $\mathrm{rk}(W_b) = d - \lambda'_m$. The rest of the proof splits into two cases:

- If $R = R^{(2,2m+1)}$, we need to show that $\mathrm{rk}_{\widetilde{A}}(\widetilde{A}W_b) = \mathrm{rk}_A(W_b)$. Indeed, since $\widetilde{A}W_b = W_b \oplus TW_b \simeq W_b \otimes_A \widetilde{A}$, and the residue maps $A \to k$ and $\widetilde{A} \to k$ are compatible with respect to the inclusion $A \to \widetilde{A}$, we have $(\widetilde{A}W_b) \otimes_{\widetilde{A}} k \simeq W_b \otimes_A k$.
- If $R = R^{(2,2m)}$, we need to show that $\mathrm{rk}_{e_i\widetilde{A}}(e_i\widetilde{A}W_b) = \mathrm{rk}_A(W_b)$ for $i = 1, 2$, where $e_i\widetilde{A}$ is a ring with identity element e_i. The desired claim follows immediately from the fact that multiplication by e_i gives a ring isomorphism $A \to e_i\widetilde{A}$. ☐

To describe $E_R(\widetilde{R}L_b; L_b)$, we apply Lemma 8.10, fixing a free R-lattice F_b such that $\widetilde{R}F_b = \widetilde{R}L_b$ if $R = R^{(2,2m+1)}$. The nature of the parametrization in Lemma 8.10 depends on the type of W'_{L_b} as a $k[[X]]$-module, which we now determine in terms of λ. We warn the reader that W'_{L_b} is entirely different from W_b: the former lives in $\widetilde{R}L_b/\mathfrak{c}L_b$, while the latter lives in $F/\mathfrak{c}F$.

Lemma 8.14. *Let $R = R^{(2,n)}$ and assume the notation above. Then the type of W'_{L_b} is λ.*

Proof. The key claim is that $\widetilde{R}L_b/L_b \simeq L_b/\mathfrak{c}F$ as $R/\mathfrak{c}$-modules. Working modulo $\mathfrak{c}F$, the claim is equivalent to $\widetilde{A}W_b/W_b \simeq W_b$ as A-modules. For $R = R^{(2,2m+1)}$, we have $\widetilde{A}W_b = W_b \oplus TW_b$, so the map $W_b \to \widetilde{A}W_b/W_b$ induced by multiplication by T gives the desired isomorphism. For $R = R^{(2,2m)}$, we have $\widetilde{A}W_b = W_b \oplus e_1 W_b$, so the map $W_b \to \widetilde{A}W_b/W_b$ induced by multiplication by e_1 gives the desired isomorphism.

Now in the classification in Lemma 8.10, we have $V_1/W'_{L_b} \simeq \widetilde{R}L_b/L_b$ as A-modules. By the above claim, the type of V_1/W'_{L_b} is the same as the type of W_b, which is $(m^d) - \lambda$. Since $V_1 \simeq A^d$, it follows that the type of W'_{L_b} is λ. ☐

We are ready to prove the "harder" part of Theorems 1.8 and 1.10 up to point counts.

Proposition 8.15. *If $R = R^{(2,2m+1)}$ and $k = \mathbb{F}_q$, then*

$$|NZ_{R^d}(t)|_q = \sum_{\mu \subseteq \lambda \subseteq (m^d)} g_\lambda^{(m^d)}(q)\, g_\mu^\lambda(q)\, (t;q)_{d-\lambda'_m} t^{|\lambda|} (q^d t)^{|\mu|}. \tag{8.20}$$

Proof. This follows from substituting Lemma 8.13, Lemma 8.5, Lemma 8.10 and Lemma 8.14 into (8.2). More precisely, we use the notation of (8.2), Lemma 8.13 and Lemma 8.14. In Lemma 8.10, we replace $\widetilde{L}$ by $\widetilde{R}L_b$. Finally, we let λ be the type of W_b and μ be the cotype of W'_L in W'_{L_b}. □

Remark 8.16. In Lemma 9.3, we will further simplify (8.20) to the form in Theorem 1.8.

Proposition 8.17. *If $R = R^{(2,2m)}$ and $k = \mathbb{F}_q$, then*

$$|NZ_{R^d}(t)|_q = \sum_{\mu\subseteq\lambda\subseteq(m^d)} g_\lambda^{(m^d)}(q)\, g_\mu^\lambda(q)\, (t;q)^2_{d-\lambda'_m} t^{|\lambda|}(q^d t)^{|\lambda|-|\mu|} \frac{(q^{-1};q^{-1})_{\lambda'_m}}{(q^{-1};q^{-1})_{\mu'_m}}. \tag{8.21}$$

Proof. The proof is analogous to the proof of Proposition 8.15, except we replace Lemma 8.5 by Lemma 8.6, apply Remark 8.11, and let μ be the type of W'_L instead. □

8.5. **The motivic version.** To finish the proof of Theorems 1.8 and 1.10, we need to show that parametrizations described above are motivic, i.e., they correspond to stratifications and fibrations. We just focus on the geometrizing Proposition 8.15; the rest is similar.

The issue arises in padding (Lemmas 4.3 and 4.4), where we make a noncanonical choice for a new object for each previously chosen object that varies in a continuum. Thanks to Remark 5.40, we can ignore this problem, i.e., it suffices to show that the family $\mathcal{Y} := \{(L, L_b) : L \in E_R(\widetilde{R}L_b; L_b), L_b \in \partial_{\mathfrak{c}F}(F)\}$ has rational motive (i.e., the motive is in $\mathbb{Z}[\mathbb{L}]$). Here the family depends on relative indices $[F : L_b]$ and $[L_b : L]$ that we implicitly fix.

Recall that we stratify the space of L_b's according to the type λ of F/L_b. Denote by $\mathcal{B}_\lambda$ the locus of L_b with a fixed λ. It clearly has rational motive. Recall that we parametrize each (L, L_b) by a triple (L_b, W'_L, φ_L), notation as in Lemma 8.10, where the space of possible (W'_L, φ_L)'s depends (through V_1, V_2, and W'_{L_b}) on F_b, a free R-lattice such that $\widetilde{R}F_b = \widetilde{L}_b$ that we have to fix noncanonically in advance. But since F_b can be obtained as the R-linear span of any basis of $\widetilde{R}L_b$, there is a stratification $\mathcal{B}_{\lambda,i}$ of $\mathcal{B}_\lambda$ and a "coherent choice" F_b in family for each $L_b \in \mathcal{B}_{\lambda,i}$. Using these, we can build a stratification $\mathcal{Y} = \bigsqcup_{\lambda,\mu,i} \mathcal{Y}_{\lambda,\mu,i}$ where $\mathcal{Y}_{\lambda,\mu,i}$ consists of elements (L_b, W'_L, φ_L) of $\mathcal{Y}$ such that $L_b \in \mathcal{B}_{\lambda,i}$ and W'_L has type μ. One can easily identify $\mathcal{Y}_{\lambda,\mu,i}$ with a trivial bundle over $\mathcal{B}_{\lambda,i}$ whose fibers are isomorphic to the space of solutions to Lemma 8.10 as if W'_{L_b} were a fixed $k[[X]]$-module of type μ. The fiber clearly depends only on λ, μ and has rational motive. Denote the fiber by $\mathcal{F}_{\lambda,\mu}$. Then

$$[\mathcal{Y}] = \sum_{\lambda,\mu,i} [\mathcal{B}_{\lambda,i}]\, [\mathcal{F}_{\lambda,\mu}] = \sum_{\lambda,\mu} [\mathcal{B}_\lambda]\, [\mathcal{F}_{\lambda,\mu}], \tag{8.22}$$

which has rational motive. This completes the proof of Theorems 1.8 (up to Lemma 9.3) and 1.10. □

9. Combinatorial aspects

In this section, we collect some formal observations about the polynomials and power series in two variables arising from $NZ_{R^d}(t)$ and $\widehat{NZ}_R(t)$, where $R = R^{(2,2m+1)}$ or $R^{(2,2m)}$. Some of these will finish the proof of Theorems 1.8, 1.13, 1.14, and 1.16. We note two key inputs: a new combinatorial identity (Lemma 9.3) proved by q-hypergeometric techniques, and the functional equation (Theorem 1.7) for $R = R^{(2,2m)}$ that amounts to a formal identity whose direct proof is so far unknown. The former is necessary to prove the $NZ_{R^d}(t) = NZ^{R}_{\widetilde{R}^d}(t^2)$ assertion for $R = R^{(2,2m+1)}$ (Theorem 1.8), see §9.2. The latter is needed to prove Theorem 1.16, see §9.5.

We remind the readers that all polynomials and series involved in this section are explicit, and their formulas can be extracted from Propositions 8.8, 8.9, 8.15, and 8.17. Since the discussion in this section only concerns formal polynomials and series, there is no harm to take the point-counting versions of these zeta functions over $\mathbb{F}_q$ and treat q as a variable.

9.1. **Basic hypergeometric series.** Our main tool to prove formal identities is the basic hypergeometric series (or q-hypergeometric series) and its transformations. We first recall the definition.

Definition 9.1. For $r, s \geq 0$, define

$$ {}_r\phi_s\begin{bmatrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{bmatrix}; q, z \Bigg] := \sum_{k=0}^{\infty} \Big((-1)^k q^{\binom{k}{2}} \Big)^{s+1-r} \frac{(a_1;q)_k \dots (a_r;q)_k}{(q;q)_k (b_1;q)_k \dots (b_s;q)_k} z^k. \tag{9.1} $$

To apply the method of q-hypergeometric series, the first step is to convert the expression in question into a q-hypergeometric series. This often involves converting a q-Pochhammer symbol such as $(a;q)_{n\pm k}$, $(aq^k;q)_n$, and $(a;q^{-1})_k$ into the form $(A;q)_k$, where A does not depend on k. The process is straightforward once the goal is clear. Some useful identities are

$$ (a;q)_{n-k} = \frac{(a;q)_n}{(aq^{n-1};q^{-1})_k}, \tag{9.2} $$

$$ (a;q)_k = (-a)^k q^{\binom{k}{2}} (a^{-1};q^{-1})_k. \tag{9.3} $$

The second step is to apply transformation identities to the q-hypergeometric series we obtain. We will refer to [23] for standard identities. We state one here.

Lemma 9.2 (Cauchy, [23, (II.5)])**.** *As formal series in a, z, q, we have*

$$ {}_1\phi_1\begin{bmatrix} a \\ az \end{bmatrix}; q, z \Bigg] := \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{(a;q)_k}{(q;q)_k (az;q)_k} z^k = \frac{(z;q)_\infty}{(az;q)_\infty}. \tag{9.4} $$

9.2. **Proof of Theorem 1.8.** To simplify (8.20), we need the following identity, which appears to be new. We thank S. Ole Warnaar for sketching its proof.

Lemma 9.3. *For $d, m \geq 0$ and any partition $\mu \subseteq (m^d)$, we have the following identity in $\mathbb{Z}[t,q]$:*

$$ \sum_{\lambda:\, \mu \subseteq \lambda \subseteq (m^d)} g_\lambda^{(m^d)}(q)\, g_\mu^\lambda(q)\, t^{|\lambda|} (t;q)_{d-\lambda'_m} = g_\mu^{(m^d)}(q)\, t^{|\mu|}. \tag{9.5} $$

Proof. We denote $(q^{-1})_n := (q^{-1};q^{-1})_n$ to save space. Applying Theorem 3.4 and dividing both sides by $q^{-\sum_{i\geq 1} \mu_i'^2} (q^{-1})_d$, the original identity is equivalent to

$$ \sum_{\lambda:\, \mu \subseteq \lambda \subseteq (m^d)} q^{-\sum_{i\geq 1} \lambda'_i(\lambda'_i - \mu'_i)} \frac{(t;q)_{d-\lambda'_m} (q^d t)^{|\lambda|}}{\prod_{i\geq 0} (q^{-1})_{\lambda'_i - \lambda'_{i+1}}} \prod_{i\geq 0} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_{q^{-1}} = \frac{(q^d t)^{|\mu|}}{\prod_{i\geq 0} (q^{-1})_{\mu'_i - \mu'_{i+1}}}, \tag{9.6} $$

where $\lambda'_0 = \mu'_0 := d$.

The strategy to prove (9.6) is as follows. We denote the left-hand side of (9.6) by $f_{\mu,m,d}(t,q)$. We first fix $\lambda'_1, \dots, \lambda'_{m-1}$ and sum over λ'_m from μ'_m to λ'_{m-1}. It turns out that the sum can be evaluated by Lemma 9.2. We now sum the result over $\lambda'_1, \dots, \lambda'_{m-1}$ with $d \geq \lambda'_1 \geq \dots \geq \lambda'_{m-1}$ and $\lambda'_i \geq \mu'_i$ for $1 \leq i \leq m-1$. It then turns out that this sum is a multiple of $f_{\overline{\mu},m-1,d}(t,q)$, where $\overline{\mu}$ is the partition with $\overline{\mu}'_i = \mu'_i$ for $1 \leq i \leq m-1$ and $\overline{\mu}'_m = 0$. The identity (9.6) then follows from induction on m.

We now carry out the computation in detail. If $m = 0$, the identity reduces to $1 = 1$. Now assume $m \geq 1$. By extracting the factors that do not depend on λ'_m, and substituting $k = \lambda'_m - \mu'_m$, the left-hand side of (9.6) equals

$$ \begin{aligned} &\frac{(q^d t)^{\mu'_m}}{(q^{-1})_{\mu'_m}} \sum_{\lambda'_1, \dots \lambda'_{m-1}} q^{-\sum_{i=1}^{m-1} \lambda'_i(\lambda'_i - \mu'_i)} \frac{(q^d t)^{\sum_{i=1}^{m-1} \lambda'_i}}{\prod_{i=0}^{m-2} (q^{-1})_{\lambda'_i - \lambda'_{i+1}}} \prod_{i=1}^{m-1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_{q^{-1}} \\ &\qquad \sum_{k=0}^{\lambda'_{m-1} - \mu'_m} q^{-k(k+\mu'_m)} \frac{(t;q)_{d-\mu'_m - k} (q^d t)^k}{(q^{-1})_{\lambda'_{m-1} - \mu'_m - k} (q^{-1})_k}. \end{aligned} \tag{9.7} $$

We reorganize the inner sum of (9.7) into a q^{-1}-hypergeometric series as follows

$$\frac{(t;q)_{d-\mu'_m}}{(q^{-1})_{\lambda'_{m-1}-\mu'_m}}\sum_{k=0}^{\lambda'_{m-1}-\mu'_m}(-1)^k q^{-\binom{k}{2}}\frac{(q^{\lambda'_{m-1}-\mu'_m};q^{-1})_k}{(q^{-1})_k(q^{d-1-\mu'_m}t;q^{-1})_k}(q^{d-1-\lambda'_{m-1}}t)^k. \tag{9.8}$$

Note that the upper bound of the sum of (9.8) may be replaced by ∞, since $(q^{\lambda'_{m-1}-\mu'_m};q^{-1})_k=0$ for $k>\lambda'_{m-1}-\mu'_m$. Applying Lemma 9.2 with $a=q^{\lambda'_{m-1}-\mu'_m}$, $z=q^{d-1-\mu'_m}t$ and reorganizing, we get

$$f_{\mu,m,d}(t,q)=(q^dt)^{\mu'_m}q^{-\mu'^2_m}\begin{bmatrix}\mu'_{m-1}\\ \mu'_m\end{bmatrix}_{q^{-1}}f_{\overline{\mu},m-1,d}(t,q). \tag{9.9}$$

The desired identity then follows from induction on m. □

Proof of Theorem 1.8. By §8.5, it suffices to prove the point-count version. For $R=R^{(2,2m+1)}$, the assertion about $NZ^{R}_{\widetilde{R}^d}(t)$ is proved in Proposition 8.8. For $NZ_{R^d}(t)$, the desired formula follows from (8.20) and Lemma 9.3. More precisely, evaluating the subsum of (8.20) associated with each fixed μ using Lemma 9.3, we get

$$|NZ_{R^d}(t)|_q=\sum_{\mu\subseteq(m^d)}g_\mu^{(m^d)}(q)\,(q^dt^2)^{|\mu|}. \tag{9.10}$$

□

Remark 9.4. The identity (9.5) can be viewed as a bounded analogue of the skew Cauchy identity for Hall–Littlewood polynomials; we thank S. Ole Warnaar for pointing it out. Indeed, the $m\to\infty$ limit of (9.5) reduces to the following identity

$$(t;z)_d\sum_\lambda P_\lambda(1,z,\dots,z^{d-1};z)\,Q'_{\lambda/\nu}(t;z)=P_\mu(1,z,\dots,z^{d-1};z) \tag{9.11}$$

where $z=q^{-1}$ and $Q'_{\lambda/\mu}$ is the modified skew Hall–Littlewood polynomial (see e.g. [62]). This in turn follows from the skew Cauchy identity

$$\sum_\lambda P_\lambda(\underline{x};z)\,Q'_{\lambda/\mu}(\underline{y};z)=P_\mu(\underline{x};z)\prod_{i,j}\frac{1}{1-x_iy_j} \tag{9.12}$$

by setting $\underline{x}=(1,z,\dots,z^{d-1})$ and $\underline{y}=t$. Thus in some sense, we can view such a summation identity over partitions bounded by (∞^d) as a (skew) Cauchy type identity, and view the (m^d) counterpart as its bounded analogue (while the (m^∞) counterpart compares to Rogers–Ramanujan identities in [34]).

9.3. **Proof of Theorems 1.13 and 1.14.** Here we apply Theorems 1.8, 1.10, and 1.12 to prove Theorems 1.13 and 1.14.

Proof. We just prove the case $R=R^{(2,2m+1)}$; the case $R=R^{(2,2m)}$ is similar. By Remark 3.5 and Theorem 1.8, we have

$$NZ_{R^d}(t)=\sum_{\mu\subseteq(m^d)}\frac{1}{a_{\mathbb{L}}(\mu)}\frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_d}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{d-\mu'_1}}(\mathbb{L}^dt)^{2|\mu|}, \tag{9.13}$$

so that

$$\widehat{NZ}_R(t)=\lim_{d\to\infty}NZ_{R^d}(\mathbb{L}^{-d}t)=\lim_{d\to\infty}\sum_{\mu_1\le m}\mathbf{1}_{\mu'_1\le d}\frac{1}{a_{\mathbb{L}}(\mu)}\frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_d}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{d-\mu'_1}}t^{2|\mu|}, \tag{9.14}$$

where $\mathbf{1}_{\mu'_1\le d}$ is 1 if $\mu'_1\le d$ and 0 otherwise. Due to the rapidly decaying factor $\mathbb{L}^{-\sum_{i=1}^m\mu'^2_i}$ coming from $1/a_{\mathbb{L}}(\mu)$, it is easy to show that one may exchange the sum and the limit, so the limit converges and we get $\widehat{NZ}_R(t)=\sum_{\mu_1\le m}t^{2|\mu|}/a_{\mathbb{L}}(\mu)$ as required. □

Remark 9.5. With little modification, the argument involving the decaying factor implies that $|\widehat{NZ}_R(t)|_q$ is an entire function in t if $q > 1$ and $R = R^{(2,2m+1)}$ or $R^{(2,2m)}$. The key estimate in the $R = R^{(2,2m)}$ case is as follows:

$$q^{\sum_{i=1}^m (\lambda'_i\mu'_i - \lambda'^2_i - \mu'^2_i)} \le q^{-\sum_{i=1}^m (\lambda'^2_i + \mu'^2_i)/2}. \tag{9.15}$$

9.4. **General discussions on special values.** The following concerns the specialization $t = 1$. Assume $k = \mathbb{F}_q$ and R satisfies $(*)$. Define

$$c_{d,R}(q) := |NZ^R_{\widetilde{R}^d}(1)|_q. \tag{9.16}$$

Lemma 9.6. *Assume $k = \mathbb{F}_q$ and R satisfies $(*)$ with branching number s and Serre invariant δ.*

(a) *If E is a torsion-free R-module of rank d, then $|NZ_E(1)|_q = c_{d,R}(q)$.*
(b) *If Ω is the dualizing module of R, and $E = \Omega^d$, then $|NZ_E(\mathbb{L}^{-d})|_q = q^{-\delta d^2} c_{d,R}(q)$.*
(c) *If R is Gorenstein, then $|\widehat{NZ}_R(1)|_q = \lim_{d\to\infty} c_{d,R}(q)/q^{\delta d^2}$ if the limit exists.*

Proof.

(a) This is just a restatement of Remark 4.9.
(b) This follows from part (a) and the functional equation (1.9) with $t = 1$.
(c) This follows from part (b) and Theorem 1.12. □

The following is essentially about specializing at $\mathbb{L} \mapsto 1$. It is not used in the rest of the paper. Let k be an algebraically closed field, and consider the ring homomorphism $\chi : K_0(\mathrm{Var}_k) \to \mathbb{Z}$ defined by the Euler characteristic in the sense of Betti cohomology or ℓ-adic cohomology.

Proposition 9.7. *Assume the setting above, and let R be a complete local k-algebra of finite type with residue field k. Let E be any finitely generated module over R, and $d \ge 0$. Then*

$$\chi(Z_{E^{\oplus d}}(t)) = \chi(Z_E(t))^d. \tag{9.17}$$

Proof. We use the theorem of Białynicki-Birula [5]. Consider the natural action of the torus $(k^\times)^d$ on $E^{\oplus d}$, which induces an action on $\mathrm{Quot}_{E^{\oplus d},n}$ for any $n \ge 0$. Pick a generic one-dimensional subtorus T in $(k^\times)^d$ that contains an element $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\lambda_i \in k^\times$ distinct. We now determine the T-fixed points of $\mathrm{Quot}_{E^{\oplus d},n}$.

Let $F \in \mathrm{Quot}_{E^{\oplus d},n}$, i.e., $F \subseteq E$ is an R-submodule of k-codimension n. For $1 \le i \le d$, consider the coordinate projection $\pi_i : E^{\oplus d} \to E^{\oplus d}$ by $\pi_i(f_1, \dots, f_d) = (0, \dots, f_i, \dots, 0)$. Define $F' = \bigoplus_{i=1}^d \pi_i(F)$. We claim that F is fixed by T if and only if $F' = F$.

The "if" direction is trivial. For the "only if" direction, let $f = (f_1, \dots, f_d)$ be an arbitrary element of F. Since F is fixed by T, we have $\lambda^j f = (\lambda_i^j f_i)_i \in F$ for $0 \le j \le d-1$. Since λ_i are distinct, the Vandermonde matrix (λ_i^j) is invertible, so for each $1 \le i \le d$, we can express $\pi_i(f)$ as a k-linear combination of $\lambda^j f$. It follows that $\pi_i(f) \in F$, and thus $F' = F$.

As a result, by setting $n_i := \dim_k \pi_i(E^{\oplus d})/\pi_i(F)$, we have a locally closed decomposition

$$(\mathrm{Quot}_{E^{\oplus d},n})^T = \bigsqcup_{\substack{n_1,\dots,n_d\ge 0\\ n_1+\dots+n_d=n}} \prod_{i=1}^d \mathrm{Quot}_{E,n_i}. \tag{9.18}$$

By [5], we have $\chi(\mathrm{Quot}_{E^{\oplus d},n}) = \chi((\mathrm{Quot}_{E^{\oplus d},n})^T)$. The desired result thus follows. □

Corollary 9.8. *Treat $NZ_{R^d}(t)$ as a formal polynomial in $t, \mathbb{L}$, where $R = R^{(2,2m+1)}$ or $R^{(2,2m)}$. Then*

$$NZ_{R^d}(t)|_{\mathbb{L}\mapsto 1} = (NZ_R(t)|_{\mathbb{L}\mapsto 1})^d = \begin{cases} (\sum_{i=0}^m t^{2i})^d, & R = R^{(2,2m+1)}; \\ (\sum_{i=0}^{2m} (-t)^i)^d, & R = R^{(2,2m)}. \end{cases} \tag{9.19}$$

9.5. **Proof of Theorem 1.16.** Let $R = R^{(2,2m)}$. Though substituting $t = 1$ into Theorem 1.14 is a natural approach to Theorem 1.16, we instead apply Lemma 9.6(c). In doing this, we are essentially taking advantage of the functional equation, and that $NZ^{\widetilde{R}}_{\widetilde{R}^d}(t)$ has a simpler formula than $NZ^{R}_{R^d}(t)$. To this end, we first compute $c_{d,R}(q)$ defined in (9.16).

Lemma 9.9. *Let $k = \mathbb{F}_q$ and $R = R^{(2,2m)}$. Then $c_{d,R}(q) = q^{md^2}$.*

Proof. By (8.14) and the definition of $c_{d,R}(q)$, it suffices to prove the identity

$$q^{md^2} = \sum_{\mu\subseteq(m^d)} g^{(m^d)}_{\mu}(q)\, q^{d|\mu|} \frac{(q^{-1}; q^{-1})_d}{(q^{-1}; q^{-1})_{d-\mu'_1}}. \tag{9.20}$$

Recall §3.2. Let D be a DVR with uniformizer π and residue field $\mathbb{F}_q$. Then the above identity follows from the fact that both sides count the number of D-linear homomorphisms $f : (D/\pi^m)^d \to (D/\pi^m)^d$. Here on the right-hand side, μ is the type of the image of f, and we apply Lemma 3.6. □

Proof of Theorem 1.16. Since $\widehat{NZ}_R(t) \in \mathbb{Z}[[\mathbb{L}^{-1}, t]]$, it suffices to prove the point-count version. This follows from Lemma 9.9, Lemma 9.6(c), and the fact that $\delta = m$ for $R^{(2,2m)}$. □

9.6. **The** $(2, 2)$ **link.** It turns out that for $R = R^{(2,2m)}$ with $m = 1$, the polynomial $NZ_{R^d}(t)$ can be simplified, and its functional equation can be directly verified. We thank S. Ole Warnaar for observing the next two lemmas and sketching their proofs. Define

$$f_{d,1}(t, q) := |NZ_{R^{(2,2)}{}^d}(t)|_q = \sum_{r\geq s\geq 0} \begin{bmatrix} d \\ r \end{bmatrix}_q \begin{bmatrix} r \\ s \end{bmatrix}_q t^r (q^d t)^{r-s} (t; q)^2_{r-s} \frac{(q^{-1}; q^{-1})_r}{(q^{-1}; q^{-1})_s}, \tag{9.21}$$

which is obtained from (1.12) with $\lambda = (1^r), \mu = (1^s)$.

Lemma 9.10. *We have*

$$f_{d,1}(t, q) = {}_2\phi_1 \left[\begin{matrix} q^d, tq^d \\ 0 \end{matrix} ; q^{-1}, tq^{-1} \right] = \sum_{r=0}^{d} (-1)^r q^{\binom{r}{2}} t^r \begin{bmatrix} d \\ r \end{bmatrix}_q (tq^{d-r+1}; q)_r. \tag{9.22}$$

Proof. Set $z = q^{-1}$. We will convert everything to base-z hypergeometric series. By setting $l = r - s$, switching the summations in (9.21), and converting the inner sum in l into hypergeometric, we get

$$f_{d,1}(t, q) = \sum_{s=0}^{d} t^{2d-s} z^{-d^2+d+ds-s} (1/t; z)^2_{d-s} \begin{bmatrix} d \\ s \end{bmatrix}_z {}_2\phi_2 \left[\begin{matrix} z^{s+1}, z^{-(d-s)} \\ tz^{1-d+s}, tz^{1-d+s} \end{matrix} ; z, t^2 z^{1-d} \right]. \tag{9.23}$$

Applying the transformation [23, (III.4)], expanding the resulting ${}_2\phi_1$ as a summation in l, resetting $r = l + s$, switching the summations, and converting the inner sum in s into hypergeometric, we get

$$\begin{aligned} f_{d,1}(t, q) &= \sum_{s=0}^{d} (-1)^{d-s} t^d z^{-\binom{d}{2}+\binom{s}{2}} (1/t; z)_{d-s} \begin{bmatrix} d \\ s \end{bmatrix}_z {}_2\phi_1 \left[\begin{matrix} tz^{-d}, z^{-(d-s)} \\ tz^{1-d+s} \end{matrix} ; z, tz \right] &(9.24)\\ &= \sum_{s=0}^{d} \sum_{l=0}^{d-s} (-1)^{d-s} t^{d+l} z^{-\binom{d}{2}+\binom{s}{2}+l} (1/t; z)_{d-s} \begin{bmatrix} d \\ s \end{bmatrix}_z \frac{(tz^{-d}, z^{-(d-s)}; z)_l}{(z, tz^{1-d+s}; z)_l} &(9.25)\\ &= \sum_{r=0}^{d} (-1)^{d-r} t^{d+r} z^{-\binom{d}{2}-dr+\binom{r+1}{2}} \frac{(1/t; z)_d (tz^{-d}; z)_r}{(tz^{1-d}; z)_r} \begin{bmatrix} d \\ r \end{bmatrix}_z {}_1\phi_1 \left[\begin{matrix} z^{-(d-r)} \\ tz^{1-d+r} \end{matrix} ; z, tz \right]. &(9.26) \end{aligned}$$

Summing the ${}_1\phi_1$ by Lemma 9.2 finally gives

$$f_{d,1}(t, q) = \sum_{r=0}^{d} (-1)^r t^r z^{\binom{r}{2}-r(d-1)} (tz^{-d}; z)_r \begin{bmatrix} d \\ r \end{bmatrix}_z = {}_2\phi_1 \left[\begin{matrix} z^{-d}, tz^{-d} \\ 0 \end{matrix} ; z, tz \right], \tag{9.27}$$

finishing the proof. The last part of (9.22) just rewrites everything with positive powers of q. □

Remark. The analogous method for $m > 1$ fails. One can reach an analogue of (9.26), except that we get a ${}_2\phi_2$ that does not simplify to a ${}_1\phi_1$ unless $\lambda'_{m-1} = \mu'_{m-1}$ (with the convention that $\lambda'_0 = \mu'_0 := d$), which is only guaranteed when $m = 1$. The ${}_2\phi_2$ is transformable but not summable.

Lemma 9.11. *We have $f_{d,1}(t, q) = q^{d^2} t^{2d} f_{d,1}(q^{-d}t^{-1}, q)$.*

Proof. This is already guaranteed by Theorem 1.7, but here is a direct proof. By [23, (III.6)], we have

$$f_{d,1}(t,q) = \lim_{c\to 0} {}_2\phi_1\left[\begin{matrix} q^d, tq^d \\ c \end{matrix}; q^{-1}, tq^{-1}\right] \tag{9.28}$$

$$= \lim_{c\to 0} \frac{(ct^{-1}z^d; z)_d}{(c; z)_d} (t^2 z^{-d})^d {}_3\phi_2\left[\begin{matrix} z^{-d}, t^{-1}, c^{-1}z^{1-d} \\ 0, c^{-1}tz^{1-2d} \end{matrix}; z, z\right], \tag{9.29}$$

where $z = q^{-1}$ as before. Using the elementary limit

$$\lim_{c\to 0} \frac{(c^{-1}a; z)_l}{(c^{-1}b; z)_l} = \left(\frac{a}{b}\right)^l \tag{9.30}$$

and the definition of basic hypergeometric series, we get as required

$$f_{d,1}(t,q) = (t^2z^{-d})^d {}_2\phi_1\left[\begin{matrix} z^{-d}, t^{-1} \\ 0 \end{matrix}; z, t^{-1}z^{1+d}\right] = (t^2z^{-d})^d f_{d,1}(t^{-1}z^d, q). \tag{9.31}$$

□

We can similarly simplify the Cohen–Lenstra zeta function. From (1.20), let

$$\widehat{f}_1(t,q) := |\widehat{NZ}_{R^{(2,2)}}(t)|_q = (tq^{-1}; tq^{-1})_\infty^2 \sum_{r\geq s\geq 0} \frac{\begin{bmatrix} r \\ s \end{bmatrix}_q t^{2r-s}}{q^{r^2}(q^{-1};q^{-1})_s (tq^{-1};q^{-1})_r^2} \in \mathbb{Z}[[t, q^{-1}]]. \tag{9.32}$$

Lemma 9.12 (cf. [22, 36])**.** *We have three equivalent formulas for $\widehat{f}_1(t;q)$:*

$$\text{(i) } {}_1\phi_1\left[\begin{matrix} t \\ 0 \end{matrix}; q^{-1}, tq^{-1}\right], \text{(ii) } (t, tq^{-1}; q^{-1})_\infty \cdot {}_2\phi_1\left[\begin{matrix} 0, 0 \\ tq^{-1} \end{matrix}; q^{-1}, t\right], \text{(iii) } (tq^{-1}; q^{-1})_\infty \cdot {}_0\phi_1\left[\begin{matrix} - \\ tq^{-1} \end{matrix}; q^{-1}, t^2q^{-1}\right]. \tag{9.33}$$

Proof. Applying Theorem 1.12 and working with the topology on $\mathbb{Z}[[t, q^{-1}]]$, we have

$$\widehat{f}_1(t,q) = \lim_{d\to\infty} f_{d,1}(tq^{-d}, q) \tag{9.34}$$

$$= \lim_{d\to\infty} {}_2\phi_1\left[\begin{matrix} q^d, t \\ 0 \end{matrix}; q^{-1}, tq^{-d-1}\right] \tag{9.35}$$

$$= \lim_{d\to\infty} \sum_{k=0}^{d} \frac{(t; q^{-1})_k}{(q^{-1}; q^{-1})_k} (q^{-d-1} - q^{-1}) \dots (q^{-d-1} - q^{-k}) t^k \tag{9.36}$$

$$= \sum_{k=0}^{\infty} \frac{(t; q^{-1})_k}{(q^{-1}; q^{-1})_k} (-1)^k q^{-\binom{k+1}{2}} t^k = {}_1\phi_1\left[\begin{matrix} t \\ 0 \end{matrix}; q^{-1}, tq^{-1}\right], \tag{9.37}$$

getting (i). The remaining two are obtained as limits of Heine ${}_2\phi_1$ transforms of (i), and we leave the details to the reader. □

Remark. The formulas (ii)(iii) for $|\widehat{NZ}_{R^{(2,2)}}(t)|_q$ were independently obtained in [22, 36].

9.7. **Further conjectures and data.** For $R = R^{(2,2m+1)}$, from Theorems 1.8 and 1.13, the coefficients of $NZ_{R^d}(t)$ and $\widehat{NZ}_R(t)$ in $\mathbb{L}, t$ are nonnegative and have well-understood combinatorial interpretations. However, the case $R = R^{(2,2m)}$ remains mysterious even in the presence of explicit formulas. For example, is there a combinatorial interpretation for the coefficients of $NZ_{R^d}(t)$ and the functional equation? The following conjecture based on numerical data might shed some light.

Conjecture 9.13 (Positivity)**.** *For $m \geq 1$, $d \geq 0$, let $R = R^{(2,2m)}$, then we have*

(a) $NZ_{R^d}(-t) \in \mathbb{N}[\mathbb{L}, t]$;
(b) $\widehat{NZ}_R(-t) \in \mathbb{N}[[\mathbb{L}^{-1}, t]]$.

The case $m = 1$ is clear, thanks to §9.6. The "$m \to \infty$" limit of the conjecture is not hard to see. In fact, an application of the skew Cauchy identity in Remark 9.4 implies

$$\lim_{m\to\infty} NZ_{(R^{(2,2m)})^d}(t) = \frac{(t;\mathbb{L})_d}{(\mathbb{L}^d t^2;\mathbb{L})_d} \in \mathbb{Z}[[\mathbb{L}, t]], \tag{9.38}$$

$$\lim_{m\to\infty} \widehat{NZ}_{R^{(2,2m)}}(t) = \frac{(\mathbb{L}^{-1}t;\mathbb{L}^{-1})_\infty}{(\mathbb{L}^{-1}t^2;\mathbb{L}^{-1})_\infty} \in \mathbb{Z}[[\mathbb{L}^{-1}, t]]. \tag{9.39}$$

Finally, we provide Tables 1–3 for $NZ_{R^d}(t)$ and $\widehat{NZ}_R(t)$ for $R = R^{(2,2m)}$. We put $NZ^R_{\widetilde{R}^d}(t)$ side by side with $NZ_{R^d}(t)$ in response to the discussions below Theorem 1.10, where the terms in $d = 1, 2$, $m = 1$ and $d = 1, m = 2$ are organized differently to suggest some vague patterns.

d	$NZ_{R^d}(t)\vert_{\mathbb{L}\mapsto q}$	$NZ^R_{\widetilde{R}^d}(t)\vert_{\mathbb{L}\mapsto q}$
1	$1 - t + qt^2$	$1 - t + qt$
2	$1-(q+1)t+(q^3+q^2+q)t^2-(q^3+q^2)t^3+q^4t^4$	$1-(q+1)t+(q^3+q^2)t+(q-q^3-q^2)t^2+q^4t^2$
3	$1-(q^2+q+1)t+(q^5+q^4+2q^3+q^2+q)t^2-(q^6+2q^5+2q^4+2q^3)t^3+(q^8+q^7+2q^6+q^5+q^4)t^4-(q^8+q^7+q^6)t^5+q^9t^6$	$1+(q^5+q^4+q^3-q^2-q-1)t+(q^8+q^7-2q^5-2q^4+q^2+q)t^2+(q^9-q^8-q^7+q^5+q^4-q^3)t^3$

TABLE 1. $NZ_{R^d}(t)$ with $R = R^{(2,2m)}, m = 1$

d	$NZ_{R^d}(t)\vert_{\mathbb{L}\mapsto q}$	$NZ^R_{\widetilde{R}^d}(t)\vert_{\mathbb{L}\mapsto q}$
1	$1 - t + qt^2 - qt^3 + q^2t^4$	$1 - t + qt - qt^2 + q^2t^2$
2	$1-(q+1)t+(q^3+q^2+q)t^2-(q^4+2q^3+q^2)t^3+(q^6+q^5+2q^4+q^3)t^4-(q^6+2q^5+q^4)t^5+(q^7+q^6+q^5)t^6-(q^7+q^6)t^7+q^8t^8$	$1+(q^3+q^2-q-1)t+(q^6+q^5-2q^3-q^2+q)t^2+(q^7-2q^5+q^3)t^3+(q^8-q^7-q^6+q^5)t^4$
3	$1-(q^2+q+1)t+(q^5+q^4+2q^3+q^2+q)t^2-(q^7+2q^6+3q^5+2q^4+2q^3)t^3+(q^{10}+q^9+3q^8+3q^7+4q^6+2q^5+q^4)t^4-(q^{11}+3q^{10}+4q^9+5q^8+3q^7+2q^6)t^5+(q^{13}+2q^{12}+4q^{11}+4q^{10}+5q^9+2q^8+q^7)t^6-(q^{14}+3q^{13}+4q^{12}+5q^{11}+3q^{10}+2q^9)t^7+(q^{16}+q^{15}+3q^{14}+3q^{13}+4q^{12}+2q^{11}+q^{10})t^8-(q^{16}+2q^{15}+3q^{14}+2q^{13}+2q^{12})t^9+(q^{17}+q^{16}+2q^{15}+q^{14}+q^{13})t^{10}-(q^{17}+q^{16}+q^{15})t^{11}+q^{18}t^{12}$	$1+(q^5+q^4+q^3-q^2-q-1)t+(q^{10}+q^9+2q^8-q^6-3q^5-2q^4+q^2+q)t^2+(q^{13}+2q^{12}+q^{11}-2q^{10}-3q^9-3q^8+2q^6+2q^5+q^4-q^3)t^3+(q^{16}+q^{15}+q^{14}-2q^{13}-3q^{12}-2q^{11}+q^{10}+3q^9+q^8-q^6)t^4+(q^{17}-q^{15}-2q^{14}+2q^{12}+q^{11}-q^9)t^5+(q^{18}-q^{17}-q^{16}+q^{14}+q^{13}-q^{12})t^6$

TABLE 2. $NZ_{R^d}(t)$ with $R = R^{(2,2m)}, m = 2$

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m	$\widehat{NZ}_{R^{(2,2m)}}(t)\|_{\mathbb{L}^{-1}\mapsto q}$
1	$1-(q+q^2+q^3+q^4+q^5+\cdots)t+(q+q^2+2q^3+2q^4+3q^5+3q^6+4q^7+4q^8+\cdots)t^2-(q^3+2q^4+3q^5+5q^6+6q^7+8q^8+10q^9+12q^{10}+\cdots)t^3+(q^4+q^5+3q^6+4q^7+7q^8+9q^9+14q^{10}+17q^{11}+\cdots)t^4-(q^7+2q^8+4q^9+7q^{10}+11q^{11}+16q^{12}+23q^{13}+31q^{14}+\cdots)t^5+\cdots$
2	$1-(q+q^2+q^3+q^4+q^5+\cdots)t+(q+q^2+2q^3+2q^4+3q^5+3q^6+4q^7+4q^8+\cdots)t^2-(q^2+2q^3+3q^4+4q^5+6q^6+7q^7+9q^8+11q^9+\cdots)t^3+(q^2+q^3+3q^4+4q^5+7q^6+9q^7+13q^8+16q^9+\cdots)t^4-(q^4+3q^5+5q^6+9q^7+13q^8+19q^9+26q^{10}+35q^{11}+\cdots)t^5+\cdots$
3	$1-(q+q^2+q^3+q^4+q^5+\cdots)t+(q+q^2+2q^3+2q^4+3q^5+3q^6+4q^7+4q^8+\cdots)t^2-(q^2+2q^3+3q^4+4q^5+6q^6+7q^7+9q^8+11q^9+\cdots)t^3+(q^2+q^3+3q^4+4q^5+7q^6+9q^7+13q^8+16q^9+\cdots)t^4-(q^3+2q^4+4q^5+6q^6+10q^7+14q^8+20q^9+27q^{10}+\cdots)t^5+\cdots$

TABLE 3. $\widehat{Z}_{R^{(2,2m)}}(t)$ with $1 \le m \le 3$

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