Spiral shifting operators from the enumeration of finite-index submodules of $\mathbb{F}_q[[T]]^d$

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Abstract

Fix an integer $d \geq 1$. We count full-rank $\mathbb{F}_q[[T]]$-submodules of $\mathbb{F}_q[[T]]^d$ of a given index and prove a formula analogous to Solomon's formula [26] over $\mathbb{Z}$. We interpret Solomon's method in our setting as a Gröbner stratification with respect to a "lex" monomial order, and we describe an alternative stratification by considering an "hlex" monomial order. The Gröbner stratification in the hlex monomial order gives rise to a "total distance" statistics W and certain "spiral shifting" operators $g_1, \ldots, g_d$ on the set $X = \mathbb{N}^d$ of d-tuples of nonnegative integers. These combinatorial structures turn out to be of independent interest beyond giving an alternative proof of Solomon's formula over $\mathbb{F}_q[[T]]$. We show that these operators commute, and collectively act on X freely and transitively, which in particular gives a nontrivial self-bijection on $\mathbb{N}^d$. We give a formula on how g_j interacts with the W-statistics on X, leading to a rational formula for the generating function for W-statistics over any forward orbit in X under these operators. Unlike the one used by Solomon, our Gröbner stratification encodes the Smith normal form, which leads to an alternative proof of a result of Petrogradsky.

1 Introduction

Counting algebraic objects such as modules and rings frequently leads to interesting zeta functions [8, 16, 19]. Counting submodules of a given module is no exception. Consider the Solomon zeta function

$$\zeta_{\mathbb{Z}}(\mathbb{Z}^d; s) = \sum_M (\mathbb{Z}^d : M)^{-s}, \tag{1.1}$$

where the sum extends over all finite-index submodules (or lattices) of $\mathbb{Z}^d$, and $(\mathbb{Z}^d : M)$ denotes the index of M. In 1977, Solomon [26] proved a beautiful formula

$$\zeta_{\mathbb{Z}}(\mathbb{Z}^d; s) = \zeta(s)\zeta(s-1) \ldots \zeta(s-d+1) \tag{1.2}$$

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using the Hermite normal form; see also four other proofs in the monograph [20]. The motivation of this paper is an analogous formula over the power series ring $\mathbb{F}_q[[T]]$:

$$\sum_M t^{\dim_{\mathbb{F}_q}(\mathbb{F}_q[[T]]^d/M)} = \frac{1}{(1-t)(1-tq)\dots(1-tq^{d-1})}, \tag{1.3}$$

where the sum extends over all finite-index submodules of $\mathbb{F}_q[[T]]^d$.

Solomon's proof applies to any Dedekind domain R with a well-defined Dedekind zeta function $\zeta_R(s)$, giving the direct analog of (1.2):

$$\zeta_R(R^d; s) = \zeta_R(s)\zeta_R(s-1)\dots\zeta_R(s-d+1), \tag{1.4}$$

and hence (1.3) as a special case. The content of this paper concerns a different proof as well as several independently interesting combinatorial objects arising from it. In order to introduce our proof, we shall first explain Solomon's proof of (1.3) in the context of Gröbner basis theory [15]. The theory of the Hermite normal form states that every finite-index submodule M of $\mathbb{F}_q[[T]]^d$ can be uniquely expressed as the column span of a matrix of the form

$$M = \operatorname{im} \begin{bmatrix} T^{n_1} & & & \\ a_{21}(T) & T^{n_2} & & \\ \vdots & \vdots & \ddots & \\ a_{d1}(T) & a_{d2}(T) & \cdots & T^{n_d} \end{bmatrix}, \tag{1.5}$$

where $n_i \geq 0$ and each $a_{ij}(T)$ is a polynomial in T of degree strictly less than n_i. (We say the zero polynomial to have degree -1.) Therfore, the set of finite-index submodules of $\mathbb{F}_q[[T]]^d$ is *stratified* into strata indexed by $n_1, \dots, n_d \geq 0$. From it, the formula (1.3) follows elementarily, as in [26]. Now, we denote by $u_1, \dots, u_d$ the standard basis vectors of $\mathbb{F}_q[[T]]^d$ and let $f_j = T^{n_j}u_j + \sum_{i=j+1}^n a_{ij}(T)u_i$. Then M is the submodule generated by f_i. Moreover, the requirement for the Hermite normal form is equivalent to saying that $(f_1, \dots, f_d)$ is a reduced Gröbner basis with respect to the monomial order $\prec_{\mathrm{lex}}$, where $t^a u_i \prec_{\mathrm{lex}} t^b u_j$ if $i < j$ or $(i = j$ and $a < b)$.

In the rest of this paper, we consider a different monomial order $\prec_{\mathrm{hlex}}$, where $t^a u_i \prec_{\mathrm{hlex}} t^b u_j$ if $a < b$ or $(a = b$ and $i < j).$[1] We give an alternative proof of (1.3) by classifying reduced Gröbner bases with respect to $\prec_{\mathrm{hlex}}$. This gives a different stratification of the set of finite-index submodules of $\mathbb{F}_q[[T]]^d$, also indexed by $n_1, \dots, n_d \geq 0$. We compute the cardinality of each stratum in terms of a statistics we denote by W (2.15). The left-hand side of (1.3) can now be expressed as a sum involving the W-statistics over all strata. To study this sum, we consider certain "spiral shifting" operators on the index set $X = \mathbb{N}^d = \{(n_1, \dots, n_d) : n_i \geq 0\}$ of the strata. We prove combinatorial properties of these operators and how they interact with the W-statistics (Lemma 5.2), which leads to our proof of (1.3).

The hlex-stratification above can be interpreted as another normal form (6.1) which, just like the Hermite normal form, classifies square matrices over $\mathbb{F}_q[[T]]$ up to column operations. However, unlike the Hermite normal form, the new normal form recovers the Smith normal form by its diagonal entries (Proposition 6.1). As an application of this normal form, we give

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[1]The order $\prec_{\mathrm{hlex}}$ is both a lexicographic order with $u_1 \prec \cdots \prec u_d \prec T$ and a homogeneous lexicographic order with $u_1 \prec \cdots \prec u_d$ (the position of T does not matter). We choose to call it "hlex" for the flavor reason that the set of monomials under $\prec_{\mathrm{hlex}}$ is order-isomorphic to $\mathbb{N}$, while it is not the case in $\prec_{\mathrm{lex}}$.

an alternative proof of a theorem of Petrogradsky [25] that counts submodules M of $\mathbb{F}_q[[T]]^d$ such that the quotient $\mathbb{F}_q[[T]]^d/M$ has a given $\mathbb{F}_q[[T]]$-module structure (a refinement of the information of cardinality). We do it by reducing the theorem to a refined sum involving the W-statistics. Our alternative proof also provides a direct explanation of a permutation sum in Petrogradsky's formula; see Remark 6.8.

Apart from their applications to module counting, the spiral shifting operators also give a nontrivial self-bijection on $\mathbb{N}^d$ (Theorem 4.2) that might lead to other combinatorial identities. We next state the combinatorial results independent of the context of submodule counting.

1.1 Main combinatorial results

Fix $d \geq 1$. We denote by $\mathbb{N}$ the set of nonnegative integers, and we use the standard notation $[d] := \{1, 2, \ldots, d\}$. Let $X = \mathbb{N}^d$ be the set of d-tuples $(n_1, \ldots, n_d)$ of nonnegative integers. For $x = (n_1, \ldots, n_d) \in X$, let $n(x) = n_1 + \cdots + n_d$ and let $W(x) \in \mathbb{N}$ be the statistics defined in Section 5. The key identity we prove about the n- and W-statistics is the following result.

Theorem 1.1 (Theorem 5.3). *As power series in $\mathbb{Z}[[t, q]]$, we have*

$$\sum_{x \in X} t^{n(x)} q^{W(x)} = \frac{1}{(1-t)(1-tq)\ldots(1-tq^{d-1})}. \tag{1.6}$$

In particular, for given $n, W \geq 0$, the number of elements x in X with $n(x) = n$ and $W(x) = W$ equals the number of size-W partitions whose Young diagram fits in an $n \times (d-1)$ rectangle.

To prove Theorem 1.1, we define self-maps $g_1, \ldots, g_d$ on X (Section 3). They satisfy the following properties. First, they commute.

Theorem 1.2 (Theorem 4.1). *For all $1 \leq j, j' \leq d$, we have*

$$g_{j'} \circ g_j = g_j \circ g_{j'} \tag{1.7}$$

as maps from X to X.

To state the next property, we note that as a result of Theorem 1.2, the operators g_j induce an action of the free abelian semigroup $\Gamma := \mathbb{N}^d$ on X. For $a = (a_1, \ldots, a_d) \in \Gamma$, the action is defined as

$$a \cdot x := g_1^{a_1} \circ \cdots \circ g_d^{a_d}(x) \tag{1.8}$$

for $x \in X$.

The next result states that the action is free and transitive.

Theorem 1.3 (Theorem 4.2). *Let $\mathbf{0} = (0, \ldots, 0) \in X$. Then for any $x \in X$, there exists a unique $a \in \Gamma$ such that*

$$a \cdot \mathbf{0} = x. \tag{1.9}$$

The reason why the operators g_j play a role in the proof of Theorem 1.1 is the following formula about how g_j interacts with the n- and W-statistics.

Theorem 1.4 (Lemma 5.1 and Lemma 5.2). *For any $x \in X$ and $j \in [d]$, we have*

$$W(g_j(x)) = W(x) + j - 1; \tag{1.10}$$
$$n(g_j(x)) = n(x) + 1. \tag{1.11}$$

The three properties above give a bijective proof of the equinumerosity statement in Theorem 1.1.

Corollary 1.5. *There is a bijection from the set of size-W partitions with at most n parts such that each part has size at most $d-1$ to the set of elements $x \in X$ with $n(x) = n$ and $W(x) = W$ given by*

$$[1]^{m_1} \dots [d-1]^{m_{d-1}} \mapsto g_1^{n - \sum_{i=1}^{d-1} m_i} g_2^{m_1} \dots g_d^{m_{d-1}}(\mathbf{0}). \tag{1.12}$$

Proof. The map is well-defined (i.e., landing in the target) due to Theorem 1.4. Its injectivity follows from the uniqueness part of Theorem 1.3. To show its surjectivity, assume $x \in X$ is such that $n(x) = n$ and $W(x) = W$. By the existence part of Theorem 1.3, we can write $x = g_1^{m_0} \dots g_d^{m_{d-1}}$ for some $m_0, \dots, m_{d-1} \geq 0$. It follows from Theorem 1.4 that $\sum_{i=0}^{d-1} m_i = n$ and $\sum_{i=1}^{d-1} i m_i = W$, which implies that $[1]^{m_1} \dots [d-1]^{m_{d-1}}$ is in the source and is mapped to x. $\square$

Theorem 1.1 can be further generalized in terms of the semigroup action (1.8) defined by the operators g_j.

Theorem 1.6 (Theorem 5.4 and Corollary 5.5). *Let Γ' be a finitely generated subsemigroup of Γ, and let x_0 be an element of X. Then $\sum_{x \in \Gamma' \cdot x_0} t^{n(x)} q^{W(x)}$ is a rational function in t and q. Moreover, if Γ' is free, then there is an explicit product formula in terms of any basis of Γ'.*

When $\Gamma' = \Gamma$ and $x_0 = \mathbf{0}$, this result recovers Theorem 1.1.

1.2 Related work

Solomon's formula (1.4) can be put into the context of the moduli space of framed modules[2] and the punctual Quot scheme. Indeed, let R be the coordinate ring of an affine variety over $\mathbb{F}_q$, and let $Y = \operatorname{Spec} R$. By passing to the quotient map, classifying finite-index submodules of R^d amounts to classifying "finite d-framed R-modules", i.e., a finite R-module N together with an R-linear surjection $R^d \twoheadrightarrow N$ ("d-framing"). The moduli space of the latter classification problem is the Quot scheme of points parametrizing 0-dimensional quotients of $\mathcal{O}_Y^d$ [13]. When $d = 1$, the above Quot scheme is the Hilbert scheme of points on Y.

Solomon's formula concerns the enumerative (or motivic) aspect of the geometric objects above. If $Y = \operatorname{Spec} R$ is an affine smooth curve over $\mathbb{F}_q$, then (1.4) for R follows from an analogous formula that computes of the motive of the punctual Quot scheme in the Grothendieck ring [22]; the motive in the Grothendieck ring is a refinement of the point count. The same work [22] also generlizes the formula to the nested Quot scheme. See also the work of Bifet [5] on the punctual Quot scheme on a smooth curve and a generalization by Bagnarol, Fantechi

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[2]See also [14, 23] for the relation to framed quiver varieties.

and Perroni [2] to the punctual Quot scheme of quotients of any vector bundle on a smooth curve.

There are a number of generating-series formulas about the moduli space of modules (with or without the framing) in the literature. For framed modules, see [11] on the Hilbert scheme of smooth surfaces and [4, 12, 21] on the Hilbert scheme of singular curves. For the moduli space of unframed modules, see formulas for smooth curves [8], smooth surfaces [6, 10], nodal singular curves [16] and the (noncommutative) quantum plane [17].

1.3 Organization

In Section 2, we give some preliminaries of Gröbner bases and describe our classification of finite-index submodules of $\mathbb{F}_q[[T]]^d$. In Section 3, we define the operators g_j. In Section 4, we prove several properties of g_j. In Section 5, we define the n- and W-statistics and prove summation formulas about them using the operators g_j. Sections 3, 4 and 5 are independent of Section 2. In Section 6, we discuss the relation between our stratification to the Smith normal form and give an alternative proof of Petrogradsky [25].

Notation and conventions

We fix $d \geq 1$ for the rest of the paper. We denote by $\mathbb{N}$ the set of nonnegative integers, and we use the standard notation $[d] := \{1, 2, \ldots, d\}$. For $d \geq 1$, let $X = \mathbb{N}^d$ be the set of d-tuples $(n_1, \ldots, n_d)$ of nonnegative integers.

2 Gröbner bases

Consider the free module $F := \mathbb{F}_q[[T]]^d$ over $\mathbb{F}_q[[T]]$, and denote by $u_1, \ldots, u_d$ the standard basis of F. A **monomial** is an element $T^n u_i$ for some $n \geq 0$ and $1 \leq i \leq d$. We recall the monomial order $\prec_{\text{hlex}}$ (denoted simply by $\prec$ from now on)

$$T^a u_i \prec T^b u_j \text{ iff } a < b \text{ or } (a = b \text{ and } i < j). \tag{2.1}$$

To list the lowest few monomials,

$$u_1 \prec u_2 \prec \cdots \prec u_d \prec t u_1 \prec \ldots t u_d \prec t^2 u_1 \prec \ldots \tag{2.2}$$

The monomials are indexed by the set $[d] \times \mathbb{N}$, via the identification

$$\mu_\delta := T^n u_i \text{ if } \delta = (i, n) \in [d] \times \mathbb{N}. \tag{2.3}$$

For any nonzero element $f = \sum_{\delta \in [d] \times \mathbb{N}} c_\delta \mu_\delta$ of F, the **leading term** and the **leading monomial** of f are defined as

$$\text{LT}(f) := c_\delta \mu_\delta, \tag{2.4}$$
$$\text{LM}(f) := \mu_\delta, \tag{2.5}$$

where μ_δ is the lowest monomial such that c_δ is nonzero, which exists due to well-orderedness. For a submodule M of F, the **leading submodule** of M is defined as

$$\text{LT}(M) := \{\text{LT}(f) : f \in M\}. \tag{2.6}$$

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We collect some standard facts about Gröbner bases over power series rings (or "standard bases"), adapted to suit our purpose here; see [3] and [15]. We use the noation $\langle f_1, \ldots, f_r \rangle$ to denote the $\mathbb{F}_q[[T]]$-submodule generated by $f_1, \ldots, f_r$ in F.

Proposition 2.1. *Let M be a finite-index submodule of F. Then*

(a) *The leading submodule of M can be uniquely expressed as the form*

$$\mathrm{LT}(M) = \langle T^{n_1} u_1, \ldots, T^{n_d} u_d \rangle \tag{2.7}$$

where $n_1, \ldots, n_d \geq 0$. Moreover, there is an isomorphism of $\mathbb{F}_q$-vector spaces

$$\frac{F}{M} \cong \frac{F}{\mathrm{LT}(M)}. \tag{2.8}$$

(b) *There is a unique generating set (called the **reduced Gröbner basis** for M)*

$$G = \{f_1, \ldots, f_d\} \tag{2.9}$$

for M such that for every $i \in [d]$, we have

 (i) $\mathrm{LT}(f_i) = T^{n_i} u_i$.
 (ii) *Every nonleading term of f_i is not divisible by $\mathrm{LT}(f_j)$ for any $j \in [d]$.*

(c) *Every tuple $\{f_1, \ldots, f_d\}$ satisfying conditions (i) and (ii) in (b) is the reduced Gröbner basis for the submodule $\langle f_1, \ldots, f_d \rangle$.*

Remark 2.2. The statement (c) is not obvious; it follows from the Buchberger criterion. A priori, not every tuple $(f_1, \ldots, f_d)$ satisfying (i)(ii) is a reduced Gröbner basis, because the leading submodule of $\langle f_1, \ldots, f_d \rangle$ may be bigger than $\langle \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_d) \rangle$. The criterion to test whether $(f_1, \ldots, f_d)$ is indeed a reduced Gröbner basis is the Buchberger criterion. In our case, for each basis element u_i, there is only one monomial (namely, $\mathrm{LT}(f_i)$) involving u_i, so the Buchberger criterion automatically passes.

With Proposition 2.1, we can now classify finite-index submodules of $\mathbb{F}_q[[T]]^d$ by dividing them into strata indexed by $x = (n_1, \ldots, n_d) \in X$. Each stratum, denoted by $\mathrm{Hilb}(x)$, is the set of all finite-index submodules M of F such that

$$\mathrm{LT}(M) = \langle T^{n_1} u_1, \ldots, T^{n_d} u_d \rangle. \tag{2.10}$$

For any M in $\mathrm{Hilb}(x)$, an $\mathbb{F}_q$-basis for $F/\mathrm{LT}(M)$ is

$$\{T^a u_i : 0 \leq a < n_i, i \in [d]\}, \tag{2.11}$$

so by Proposition 2.1(a),

$$\dim_{\mathbb{F}_q} F/M = \dim_{\mathbb{F}_q} F/\mathrm{LT}(M) = n_1 + \cdots + n_d. \tag{2.12}$$

By the conditions (b)(i)(ii) in Proposition 2.1, a typical element of $\mathrm{Hilb}(x)$ is given by the reduced Gröbner basis $(f_1, \ldots, f_d)$, where

$$f_i = T^{n_i} u_i + (\text{linear combination of } T^a u_j \text{ such that } a < n_j \text{ and } T^a u_j \succ T^{n_i} u_i) \tag{2.13}$$

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Given $i \neq j \in [d]$, we notice that in order for f_i to have a term involving $T^a u_j$ for some a, we must have $T^{n_j} u_j \succ T^{n_i} u_i$, namely, $n_j + j/n > n_i + i/n$. Moreover, in this case, the number of monomials of the form $T^a u_j$ that could appear in g_i is

$$\lfloor n_j + \frac{j}{n} - n_i - \frac{i}{n} \rfloor. \tag{2.14}$$

Since the coefficient of each monomial that could appear in the nonleading terms of g_i can be chosen freely and independently, it follows from the discussion above that the cardinality of $\mathrm{Hilb}(x)$ is given by $q^{W(x)}$, where

$$W(x) = \sum_{i \neq j \in [d]} \max\{0, \lfloor n_j + \frac{j}{n} - n_i - \frac{i}{n} \rfloor\}. \tag{2.15}$$

Define $n(x) = n_1 + \cdots + n_d$ for $x = (n_1, \ldots, n_d) \in X$. We have just proved the following combinatorial formula for the left-hand side of (1.3).

Lemma 2.3. *Using the definition of $W(x)$ and $n(x)$ above, we have*

$$\sum_M t^{\dim_{\mathbb{F}_q}(\mathbb{F}_q[[T]]^d/M)} = \sum_{x \in X} q^{W(x)} t^{n(x)}, \tag{2.16}$$

where the sum on the left-hand side extends over all finite-index submodules of $\mathbb{F}_q[[T]]^d$.

Together with Theorem 1.1, the formula (1.3) follows.

3 Spiral shifting operators

Recall that $X = \mathbb{N}^d$ be the set of d-tuples $(n_1, \ldots, n_d)$ of nonnegative integers, and the goal of this section is to define the operators $g_1, \ldots, g_d : X \to X$. The operator g_1 is defined by

$$g_1(n_1, \ldots, n_d) = (n_d + 1, n_1, n_2, \ldots, n_{d-1}). \tag{3.1}$$

In order to define the other operators $g_2, \ldots, g_d$, it is helpful to view an element $x = (n_1, \ldots, n_d) \in X$ as a configuration of d points

$$\Delta(x) := \{(1, x_1), (2, x_2), \ldots, (d, x_d)\} \subseteq [d] \times \mathbb{N}. \tag{3.2}$$

For a point $\delta = (i, n)$ in $[d] \times \mathbb{N}$, we define the **seat**, the **level** and the **height** of δ to be

$$i(\delta) := i, \tag{3.3}$$
$$n(\delta) := n, \tag{3.4}$$
$$h(\delta) := n + \frac{i}{d}. \tag{3.5}$$

We linearly order the set $[d] \times \mathbb{N}$ according to the height:

$$\delta_1 \prec \delta_2 \iff h(\delta_1) < h(\delta_2). \tag{3.6}$$

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For any $s \in \mathbb{N}$ and $\delta \in [d] \times \mathbb{N}$, we define

$$\delta + s/d \tag{3.7}$$

to be the unique element of $[d] \times \mathbb{N}$ with height $h(\delta) + s/d$.

In particular, $(i,n) + 1/d = (i+1,n)$ if $1 \le i \le d-1$, and $(d,n) + 1/d = (1, n+1)$. Essentially, we are arranging the points of $[d] \times \mathbb{N}$ on an upright cylinder with a *spiral* tracing the points in the height order, so the operator $+s/d$ means shifting upwards following the spiral by s steps. Note also that $(i,n) + 1 = (i,n) + d/d = (i, n+1)$, hence the notation.

Now the operator g_1 can be alternatively understood as shifting every point upwards by one step:

$$g_1(x) := x' \text{ such that } \Delta(x') = \{\delta + 1/d : \delta \in \Delta(x)\}. \tag{3.8}$$

To define the operators $g_2, \ldots, g_d$, we first list the points of $\Delta(x)$ in the height order:

$$\Delta(x) = \{\delta^1(x) \prec \delta^2(x) \prec \cdots \prec \delta^d(x)\}. \tag{3.9}$$

As an informal definition of g_j, the operator g_j keeps $\delta^1(x), \ldots, \delta^{j-1}(x)$ fixed, and simultaneously shifts $\delta^j(x), \ldots, \delta^d(x)$ upwards to the next available seat (namely, a seat not occupied by the fixed points). More formally, define $g_j(x) = x'$ such that $\Delta(x') = \{\delta^k(x') : k \in [d]\}$, where

$$\delta^k(x') = \begin{cases} \delta^k(x), & k < j; \\ \delta^k(x) + s_k/d, & k \ge j, \end{cases} \tag{3.10}$$

where s_k is the minimal positive integer such that

$$i(\delta^k + s_k/d) \in \{i(\delta^j(x)), \ldots, i(\delta^d(x))\}. \tag{3.11}$$

Implicit in the definition is an important observation that g_j preserves the height ranking among the points, i.e., we do have $\delta^1(x') \prec \cdots \prec \delta^d(x')$ if $\delta^k(x')$ is given by definition (3.10). Indeed, when going from x to x', the lowest $j-1$ points are fixed, and the remaining points are raised in a way that preserves the relative order within.

Note the subtlety that the definition of g_j involves an interplay between the height-rankings and the seats of the points.

Example 3.1. We use the example with $d = 5, x = (n_1, \ldots, n_5) = (0, 2, 1, 0, 1) \in X = \mathbb{N}^d$ and $j = 3$ to illustrate the definition of $g_j(x)$.

First, we plot x on $\mathbb{R}^2$ by collecting the points (i, n_i):

$$\Delta(x) := \{(1,0), (2,2), (3,1), (4,0), (5,1)\}. \tag{3.12}$$

Sorting the points by the height order, we have

$$\Delta(x) = \{\delta^1(x) \prec \cdots \prec \delta^5(x)\} = \{(1,0) \prec (4,0) \prec (3,1) \prec (5,1) \prec (2,2)\}. \tag{3.13}$$

We now describe $x' := g_3(x)$ by specifying how each point of $\Delta(x)$ is moved. By definition, g_3 leaves the two lowest points intact. In this case, we have

$$\delta^1(x') = (1,0), \delta^2(x') = (4,0). \tag{3.14}$$

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For the remaining points, we sort them back according to the seat:

$$\delta^5(x) = (2,2), \delta^3(x) = (3,1), \delta^4(x) = (5,1) \tag{3.15}$$

We say the seats $2, 3, 5$ to be "available". The operator g_3 moves each of the points horizontally to the next available seat to the right, except the rightmost seat among the available seats:

$$\delta^5(x') = (3,2), \delta^3(x') = (5,1). \tag{3.16}$$

For the rightmost available seat, we move the point to the leftmost available seat and *raise the level* by 1.

$$\delta^4(x') = (2, 1+1) = (2,2). \tag{3.17}$$

Collecting the point, we have

$$\{\delta^1(x') \prec \cdots \prec \delta^5(x')\} = \{(1,0) \prec (4,0) \prec (5,1) \prec (2,2) \prec (3,2)\}, \tag{3.18}$$

namely,

$$x' = g_3(x) = (0, 2, 2, 0, 1). \tag{3.19}$$

We leave it to the readers to verify the following commutative diagram:

$$
\begin{array}{ccc}
x = (0,2,1,0,1) & \overset{g_3}{\longmapsto} & (0,2,2,0,1) \\
{\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle g_2} \\
(0,2,2,1,0) & \overset{g_3}{\longmapsto} & (0,2,2,2,0)
\end{array}
\tag{3.20}
$$

4 Properties of the operators

The first property of the operators $g_1, \ldots, g_d$ is that they commute.

Theorem 4.1. *For all $x \in X$ and $j' < j$, we have*

$$g_{j'}(g_j(x)) = g_j(g_{j'}(x)). \tag{4.1}$$

Proof. Because both $g_{j'}$ and g_j leave the lowest $j'-1$ points $\delta^1, \ldots, \delta^{j'-1}$ intact, we may remove these points and consider the remaining points only. Effectively, we are assuming $j' = 1$ without loss of generality.

Let $k = j - 1$ and $l = d - k$. Say the lowest k points of $\Delta(x)$ are

$$\{\delta^1(x), \ldots, \delta^k(x)\} = \{(p_1, m_1), \ldots, (p_k, m_k)\}, \tag{4.2}$$

where $p_1 < \cdots < p_k$. (That is, we sort these points by their seats.) Let us also name the highest l points of x as

$$\{\delta^j(x), \ldots, \delta^d(x)\} = \{(q_1, n_1), \ldots, (q_l, n_l)\}, \tag{4.3}$$

where $q_1 < \cdots < q_l$. We shall prove that $g_1(g_j(x)) = g_j(g_1(x))$ by describing both sides explicitly. As a convenient convention, we define $(d+1, n)$ to be the point $(1, n+1)$ in $[d] \times \mathbb{N}$.

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By definition, we have

$$\Delta(g_j(x)) = \{(p_1, m_1), \dots, (p_k, m_k), (q_1, n_l + 1), (q_2, n_1), \dots, (q_l, n_{l-1})\} \tag{4.4}$$

and

$$\Delta(g_1(x)) = \{(p_1 + 1, m_1), \dots, (p_k + 1, m_k), (q_1 + 1, n_1), \dots, (q_l + 1, n_l)\}, \tag{4.5}$$

where the convention $(d + 1, n) = (1, n + 1)$ applies to exactly one of points in the list for $\Delta(g_1(x))$. Applying g_1 to $\Delta(g_j(x))$, we have

$$\Delta(g_1(g_j(x))) = \{(p_1 + 1, m_1), \dots, (p_k + 1, m_k), (q_1 + 1, n_l + 1), (q_2 + 1, n_1), \dots, (q_l + 1, n_{l-1})\}. \tag{4.6}$$

Now we apply g_j to $\Delta(g_1(x))$. Notice that g_1 does not change the height ranking within the points, because it raises the height of every point by exactly $1/d$. Thus $(p_1 + 1, m_1), \dots, (p_k + 1, m_k)$ continue to be the lowest k points of $\Delta(g_1(x))$, so g_j will shift $(q_1 + 1, n_1), \dots, (q_l + 1, n_l)$. We need to consider two cases:

(a) If $q_l \neq d$, then $(q_1 + 1, n_1), \dots, (q_l + 1, n_l)$ are their honest coordinates, so shifting by g_j results in $(q_1 + 1, n_l + 1), (q_2 + 1, n_1), \dots, (q_l + 1, n_{l-1})$.

(b) If $q_l = d$, then the honest coordinates of $(q_1 + 1, n_1), \dots, (q_l + 1, n_l)$ are $(1, n_l + 1), (q_1 + 1, n_1), \dots, (q_{l-1} + 1, n_{l-1})$, so shifting by g_j results in $(1, n_{l-1} + 1), (q_1 + 1, n_l + 1), \dots, (q_{l-1} + 1, n_{l-2})$. But note that $(1, n_{l-1} + 1)$ can be represented by $(d + 1, n_{l-1}) = (q_l + 1, n_{l-1})$ by our convention.

We see that in both cases, we have

$$\Delta(g_j(g_1(x))) = \{(p_1 + 1, m_1), \dots, (p_k + 1, m_k), (q_1 + 1, n_l + 1), (q_2 + 1, n_1), \dots, (q_l + 1, n_{l-1})\}, \tag{4.7}$$

mathcing the expression of $\Delta(g_1(g_j(x)))$. Hence, we have proved the desired equality $g_1(g_j(x)) = g_j(g_1(x))$. $\qquad\square$

For a multi-index $a = (a_1, a_2, \dots, a_d)$ in $\mathbb{N}^d$, define

$$g^a(x) := g_1^{a_1} g_2^{a_2} \dots g_d^{a_d}(x). \tag{4.8}$$

By commutativity (Theorem 4.1), we have

$$g^{a+b} = g^a \circ g^b \text{ for } a, b \in \mathbb{N}^d. \tag{4.9}$$

In other words, the free abelian semigroup $\Gamma := \mathbb{N}^d$ acts on the set $X = \mathbb{N}^d$ via

$$a \cdot x = g^a(x). \tag{4.10}$$

The second property of the operators $g_1, \dots, g_d$ is that the semigroup action defined above is free and transitive, in the following sense.

Theorem 4.2. *Consider the configuration $\mathbf{0} = (0, \dots, 0)$ in $X = \mathbb{N}^d$. Then for any configuration $x \in X$, there exists a unique $a \in \Gamma = \mathbb{N}^d$ such that*

$$g^a(\mathbf{0}) = x. \tag{4.11}$$

Equivalently, the map $a \mapsto g^a(\mathbf{0})$ is a bijection from $\Gamma = \mathbb{N}^d$ to $X = \mathbb{N}^d$.

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The proof requires the following lemma.

Lemma 4.3. *For $2 \leq r \leq d$, say a configuration $x \in X$ to be r-**tight** if $\delta^r(x)$ is the lowest point in*

$$\{i(\delta^r(x)), \ldots, i(\delta^d(x))\} \times \mathbb{N} \tag{4.12}$$

that is higher than $\delta^{r-1}(x)$. Then for any $1 \leq j \leq r-1$, the configuration $g_j(x)$ is r-tight whenever x is r-tight.

Proof of Lemma 4.3. Notice that the lowest $j-1$ points of x are fixed by g_j and are not involved in $\{i(\delta^r(x)), \ldots, i(\delta^d(x))\} \times \mathbb{N}$. By removing the lowest $j-1$ points of x and reindexing the remaining seats, we may assume $j=1$. Then g_1 spiral shifts every point up by one step (in the sense of definition (3.7) where $s=1$), so the r-tightness is preserved. □

Proof of Theorem 4.2. Because g_i's commute with each other, we may choose to set up the equation $g^a(\mathbf{0}) = x$ in the form

$$g_d^{a_d} \ldots g_1^{a_1}(\mathbf{0}) = x. \tag{4.13}$$

Look at the lowest point δ^1. Since the only operator that moves δ^1 is g_1, we must have

$$\delta^1(g_1^{a_1}(\mathbf{0})) = \delta^1(x), \tag{4.14}$$

which determines a_1 uniquely. In fact, we have

$$a_1/d = h(\delta^1(x)) - h(\delta^1(\mathbf{0})), \tag{4.15}$$

since g_1 increases the height of every point by $1/d$.

Having determined $a_1, \ldots, a_{r-1}$ ($2 \leq r \leq d$), we now determine a_r by looking at the r-th lowest point δ^r. Let

$$x' = g_{r-1}^{a_{r-1}} \ldots g_1^{a_1}(\mathbf{0}), \tag{4.16}$$

and consider the equation

$$g_d^{a_d} \ldots g_r^{a_r}(x') = x. \tag{4.17}$$

The only operator among $g_d, \ldots, g_r$ that moves δ^r is g_r, so we must have

$$\delta^r(g_r^{a_r}(x')) = \delta^r(x). \tag{4.18}$$

It remains to show that there is a unique integer $a_r \geq 0$ satisfying the equation above. The uniqueness is clear because every application of g_r increases the height of δ^r strictly (see (3.10)). It suffices to prove the existence. The key observation is that $\delta^r(x) \succeq \delta^r(x')$. We now prove this observation.

Note that $\mathbf{0}$ is r-tight in the sense of Lemma 4.3. Since $x' = g_{r-1}^{a_{r-1}} \ldots g_1^{a_1}(\mathbf{0})$, by repetitively applying Lemma 4.3, we have that x' is also r-tight.

By construction, $\delta^j(x') = \delta^j(x)$ for all $1 \leq j \leq r-1$. Thus, the sets $\{\delta^r(x'), \ldots, \delta^d(x')\}$ and $\{\delta^r(x), \ldots, \delta^d(x)\}$ are the same (denoted by K), because they are both the complement of $\{\delta^1(x), \ldots, \delta^{r-1}(x)\}$. Since x' is r-tight, the point $\delta^r(x')$ is the lowest point of $K \times \mathbb{N}$ that is higher than $\delta^{r-1}(x')$. Since $\delta^r(x) \succ \delta^{r-1}(x) = \delta^{r-1}(x')$ and the seat of $\delta^r(x)$ also lies in K, we have $\delta^r(x) \succeq \delta^r(x')$, proving the desired observation.

Now, we recall from the definition (3.10) that for any configuration y such that $\{\delta^r(y), \ldots, \delta^d(y)\} = K$, the r-th lowest point $\delta^r(g_r(y))$ of $g_r(y)$ is defined to be the successor of $\delta^r(y)$ in

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$K \times \mathbb{N}$. Applying the above fact to $y = x', g_r(x'), g_r^2(x'), \ldots$, we see that $\delta^r(g_r^{a_r}(x'))$ ranges over all points of $K \times \mathbb{N}$ that are higher than or equal to $\delta^r(x')$. Together with the observation $\delta^r(x) \succeq \delta^r(x')$ that we have just proved, it follows that there exists $a_r \geq 0$ such that $\delta^r(g_r^{a_r}(x')) = \delta^r(x)$. $\qquad \square$

Corollary 4.4. *The group Γ acts on any configuration $x \in X$ freely, namely, if $g^a(x) = g^b(x)$, then $a = b$.*

Proof. By the existence part of Theorem 4.2, there is $c \in \Gamma$ such that $x = g^c(\mathbf{0})$. Hence

$$g^{a+c}(x) = g^{b+c}(x). \tag{4.19}$$

By the uniqueness part of Theorem 4.2, we have $a + c = b + c$. Since the semigroup $\Gamma = \mathbb{N}^d$ is cancellative, we have $a = b$. $\qquad \square$

5   Some combinatorial statistics

For a configuration $x = (n_1, \ldots, n_d)$ in X, we define its **size** to be

$$n(x) := n_1 + \cdots + n_d. \tag{5.1}$$

Lemma 5.1. *For any $x \in X$ and $j \in [d]$, we have*

$$n(g_j(x)) = n(x) + 1. \tag{5.2}$$

Proof. When we apply g_j to x, the rightmost point (i.e., the point with the largest seat) among $\delta^j(x), \ldots, \delta^d(x)$ loops around and increases the level by 1, while the levels of other points are unchanged. $\qquad \square$

We now introduce a more interesting statistics. For two points $\delta_1 \prec \delta_2$ in $[d] \times \mathbb{N}$, we define their **distance** to be

$$b(\delta_1, \delta_2) := \lfloor h(\delta_2) - h(\delta_1) \rfloor. \tag{5.3}$$

For a configuration $x \in X$ consisting of points $\Delta(x) = \{\delta^1(x) \prec \ldots \delta^d(x)\}$, we define the **weight** of x to be

$$W(x) := \sum_{j<k} b(\delta^j(x), \delta^k(x)). \tag{5.4}$$

We note that this definition of $W(x)$ is equivalent to (2.15). The statistics turns out to interact consistently with the operators g_j.

Lemma 5.2. *For any $x \in X$ and $j \in [d]$, we have*

$$W(g_j(x)) = W(x) + j - 1. \tag{5.5}$$

Proof. We refer to the points in

$$L := \{\delta^1(x), \ldots, \delta^{j-1}(x)\} \tag{5.6}$$

as low points, and the points in

$$H := \{\delta^j(x), \dots, \delta^d(x)\} \tag{5.7}$$

as high points. The weight of x consists of three parts: distances within L, distances within H, and distances from points of L to points of H. When applying g_j, every distance within L does not change (because every point of L is fixed). We claim that every distance within H does not change as well. To prove the claim, note that when defining distances between two points of H, we may ignore the seats of L and reindex the seats of H by $1, \dots, d-j+1$. Thus, we may assume $j = 1$ without loss of generality. In this case, g_1 increases the height of every point by $1/d$, so the distance is unchanged from definition (5.3).

Therefore, the weight difference $W(g_j(x)) - W(x)$ is solely contributed by the distances from low points to high points. Let $\alpha \in L$ and $\beta \in H$. Then we note as an important observation that while β is shifted upward following the spiral, the distance $b(\alpha, \beta)$ increases by 1 every time the seat of β passes through the seat of α. More formally, for any $\alpha \prec \beta \prec \beta' \in [d] \times \mathbb{N}$, we have

$$b(\alpha, \beta') - b(\alpha, \beta) = \#\Big((\beta, \beta'] \cap (\{i(\alpha)\} \times \mathbb{N}) \Big), \tag{5.8}$$

where $(\beta, \beta'] \cap (\{i(\alpha)\} \times \mathbb{N})$ is the set of all points δ with the same seat as α such that $\beta \prec \delta \preceq \beta'$.

Sort the seats of high points in ascending order:

$$i(H) = \{i_1 < \cdots < i_{d-j+1}\}. \tag{5.9}$$

Let β_k denote the point of H seated at i_k. Let β_k' denote the result of β_k after a shift by g_j. Then the seat of β_k' is i_{k+1} if $k < d-j+1$, and i_1 if $k = d-j+1$.

Partition the set of seats of points in L by intervals

$$I_1 = (i_1, i_2), \dots, I_{d-j} = (i_{d-j}, i_{d-j+1}), I_{d-j+1} = (i_{d-j+1}, d] \cup [1, i_1). \tag{5.10}$$

Denote by L_k the set of low points seated in I_k. Note that upon applying g_j, the point β_k is "shifted across" the interval I_k. The discussions so far imply that

$$b(\alpha, \beta_k') - b(\alpha, \beta_k) = \begin{cases} 1, & \alpha \in I_k; \\ 0, & \alpha \in L \setminus I_k. \end{cases} \tag{5.11}$$

It follows that

$$W(g_j(x)) - W(x) = \sum_{\substack{\alpha \in L \\ 1 \le k \le d-j+1}} b(\alpha, \beta_k') - b(\alpha, \beta_k) \tag{5.12}$$

$$= \sum_{1 \le k \le d-j+1} |I_k| \tag{5.13}$$

$$= |L| = j - 1, \tag{5.14}$$

as desired. $\qquad\square$

We are now ready to prove Theorem 1.1. For $x \in X$, we define the **content** of x to be the monomial

$$\mathrm{Cont}(x) := t^{n(x)} q^{W(x)} \in \mathbb{Z}[[t, q]] \tag{5.15}$$

in formal variables t and q.

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Theorem 5.3. *We have an identity of power series in $\mathbb{Z}[[t,q]]$:*

$$\sum_{x \in X} \mathrm{Cont}(x) = \frac{1}{(1-t)(1-tq)\dots(1-tq^{d-1})}. \tag{5.16}$$

In particular, the number of configurations x with $n(x) = n$ and $W(x) = W$ equals the number of size-W partitions of length at most n with parts at most $d-1$.

Proof. By Theorem 4.2, we have

$$\sum_{x \in X} \mathrm{Cont}(x) = \sum_{a \in \Gamma} \mathrm{Cont}(g^a(\mathbf{0})). \tag{5.17}$$

By Lemmas 5.1 and 5.2 and noting that $\mathrm{Cont}(\mathbf{0}) = 1$, we have

$$\sum_{x \in X} \mathrm{Cont}(x) = \sum_{a_1,\dots,a_d \geq 0} g_1^{a_1} \dots g_d^{a_d}(\mathbf{0}) \tag{5.18}$$

$$= \sum_{a_1,\dots,a_d \geq 0} t^{a_1 + \dots + a_d} q^{0a_1 + 1a_2 + \dots + (d-1)a_d} \tag{5.19}$$

$$= \frac{1}{(1-t)(1-tq)\dots(1-tq^{d-1})}. \tag{5.20}$$

The final assertion of the theorem follows from the elementary identity

$$\sum_{n=0}^{\infty} \sum_{\lambda \subseteq [d-1] \times [n]} t^n q^{|\lambda|} = \frac{1}{(1-t)(1-tq)\dots(1-tq^{d-1})}, \tag{5.21}$$

where the inner sum extends over all partitions of length at most n with parts of size at most $d-1$, and $|\lambda|$ denotes the size of λ. $\qquad\square$

We note the following alternative proof of Theorem 5.3 that uses only g_1.

Alternative proof of Theorem 5.3. Since we will vary d, we write X as X_d. Observe from (3.1) that the map $g_1 : X_d \to X_d$ is injective with image being the complement of

$$0 \times X_{d-1} := \{(0, n_2, \dots, n_d) \in X\}. \tag{5.22}$$

For any $x \in X_d$, we note that $n(g_1(x)) = n(x) + 1$ and $W(g_1(x)) = W(x)$, the latter being because g_1 preserves the distance $b(\delta^j(x), \delta^k(x))$ for every $j < k$.

We notice that for $x = (0, x') \in 0 \times X_{d-1}$, we have $n(x) = n(x')$ and $W(x) = W(x') + n(x')$, the latter being because the distance from the point $(1, 0)$ to (i, n_i) is n_i.

Let $f_d(t, q) := \sum_{x \in X_d} \mathrm{Cont}(x)$. Then we have a recurrence

$$f_d(t,q) = \sum_{x \in X_d} \mathrm{Cont}(g_1(x)) + \sum_{x' \in X_{d-1}} \mathrm{Cont}((0, x')) \tag{5.23}$$

$$= \sum_{x \in X_d} t^{n(x)+1} q^{W(x)} + \sum_{x' \in X_{d-1}} t^{n(x')} q^{W(x')+n(x')} \tag{5.24}$$

$$= t f_d(t, q) + f_{d-1}(tq, q). \tag{5.25}$$

Hence

$$f_d(t, q) = \frac{1}{1 - t} f_{d-1}(tq, q). \tag{5.26}$$

Since it is vacuously true that $f_0(t, q) = 1$, by applying (5.26) repetitvely, we have $f_1(t, q) = (1 - t)^{-1}$, $f_2(t, q) = (1 - t)^{-1} f_1(tq, q) = (1 - t)^{-1}(1 - tq)^{-1}$, ..., $f_d(t, q) = (1 - t)^{-1}(1 - tq)^{-1} \dots (1 - tq^{d-1})^{-1}$. $\square$

Compared to the alternative proof, the *current* proof of Theorem 5.3 has the advantages that it suffices to work on a fixed d, and that the product form of the generating function is manifest. Furthermore, a small modification of this proof gives a summation formula over the forward orbit of any element in X under any finitely generated subsemigroup of Γ.

Theorem 5.4. *Let x_0 be a configuration in X and Γ' be a subsemigroup of Γ freely generated by $a^{(1)}, \dots, a^{(r)}$. Then*

$$\sum_{x \in \Gamma' \cdot x_0} \mathrm{Cont}(x) = \mathrm{Cont}(x_0) \prod_{i=1}^{r} \frac{1}{1 - \mathrm{Cont}(a^{(i)})}, \tag{5.27}$$

where the content of an element $a = (a_1, \dots, a_d)$ of Γ is defined as

$$\mathrm{Cont}(a) = t^{a_1 + \dots + a_d} q^{0a_1 + 1a_2 + \dots + (d-1)a_d}. \tag{5.28}$$

Proof. Corollary 4.4 implies

$$\sum_{x \in \Gamma' \cdot x_0} \mathrm{Cont}(x) = \sum_{a \in \Gamma'} \mathrm{Cont}(g^a(x_0)). \tag{5.29}$$

Note from Lemma 5.1 and Lemma 5.2 that

$$\mathrm{Cont}(g^a(x)) = \mathrm{Cont}(a) \, \mathrm{Cont}(x) \tag{5.30}$$

for all $a \in \Gamma$ and $x \in X$. Note also that $\mathrm{Cont}(a + b) = \mathrm{Cont}(a) \, \mathrm{Cont}(b)$ for $a, b \in \Gamma$.

Since elements of Γ' can be traversed by the set of integers $k_1, \dots, k_r \geq 0$ by setting

$$a = k_1 a^{(1)} + \dots + k_r a^{(r)}, \tag{5.31}$$

we have

$$\sum_{x \in \Gamma' \cdot x_0} \mathrm{Cont}(x) = \sum_{k_1, \dots, k_r \geq 0} \mathrm{Cont}(g^{k_1 a^{(1)} + \dots + k_r a^{(r)}} x_0) \tag{5.32}$$

$$= \sum_{k_1, \dots, k_r \geq 0} \mathrm{Cont}(k_1 a^{(1)} + \dots + k_r a^{(r)}) \, \mathrm{Cont}(x_0) \tag{5.33}$$

$$= \mathrm{Cont}(x_0) \sum_{k_1, \dots, k_r \geq 0} \mathrm{Cont}(a^{(1)})^{k_1} \dots \mathrm{Cont}(a^{(r)})^{k_r} \tag{5.34}$$

$$= \mathrm{Cont}(x_0) \prod_{i=1}^{r} \sum_{k_i \geq 0} \mathrm{Cont}(a^{(i)})^{k_i} \tag{5.35}$$

$$= \mathrm{Cont}(x_0) \prod_{i=1}^{r} \frac{1}{1 - \mathrm{Cont}(a^{(i)})}. \tag{5.36}$$

$\square$

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Corollary 5.5. *Let x_0 be a configuration in X and Γ' be a finitely generated subsemigroup of Γ. Then $\sum_{x \in \Gamma' \cdot x_0} \mathrm{Cont}(x)$ is a rational function in t and q.*

Proof. By a theorem of Ito [18, Theorem 2], any finitely generated subsemigroup of $\Gamma = \mathbb{N}^d$ is a disjoint union of finitely many cosets of (possibly trivial) free subsemigroups of Γ. Hence, there are free subsemigroups $\Gamma^1, \ldots, \Gamma^r$ of Γ and elements $a^1, \ldots, a^r$ of Γ such that

$$\Gamma' = \bigsqcup_{i=1}^{r} a^i \Gamma^i. \tag{5.37}$$

It follows that

$$\sum_{x \in \Gamma' \cdot x_0} \mathrm{Cont}(x) = \sum_{i=1}^{r} \sum_{x \in \Gamma^i \cdot (a^i x_0)} \mathrm{Cont}(x), \tag{5.38}$$

where the right-hand side is a finite sum of rational functions by Theorem 5.4. $\qquad \square$

6   Relation to Smith normal form

We may translate the language of hlex Gröbner bases back to a matrix normal form that is different from the Hermite normal form (1.5). In particular, the general form of the reduced Gröbner basis described in Section 2 implies that every finite-index submodule M of $\mathbb{F}_q[[t]]^d$ can be uniquely expressed as the column span of a matrix of the form

$$M = \mathrm{im} \begin{bmatrix} T^{n_1} & a_{12}(T) & \cdots & a_{1d}(T) \\ a_{21}(T) & T^{n_2} & \cdots & a_{2d}(T) \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}(T) & a_{d2}(T) & \cdots & T^{n_d} \end{bmatrix}, \tag{6.1}$$

where

$$a_{ij}(T) \in \begin{cases} \mathrm{span}\{T^b : n_j < b < n_i\}, & \text{if } i < j; \\ \mathrm{span}\{T^b : n_j \leq b < n_i\}, & \text{if } i > j. \end{cases} \tag{6.2}$$

A special property of this new normal form is that the diagonal entries recover the Smith normal form, or equivalently, the $\mathbb{F}_q[[T]]$-module structure of $\mathbb{F}_q[[T]]^d / M$. This is not the case for the Hermite normal form. For example, consider

$$A = \begin{bmatrix} T & 0 \\ 1 & T \end{bmatrix}. \tag{6.3}$$

Then the Smith normal form of A is $\mathrm{diag}(1, T^2)$, not $\mathrm{diag}(T, T)$.

Proposition 6.1. *Let M be of the form* (6.1). *Then as an $\mathbb{F}_q[[T]]$-module, we have*

$$\frac{\mathbb{F}_q[[T]]^d}{M} \cong \frac{\mathbb{F}_q[[T]]}{T^{n_1}} \oplus \cdots \oplus \frac{\mathbb{F}_q[[T]]}{T^{n_d}}. \tag{6.4}$$

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Proof. Recall that to obtain the Smith normal form, we are allowed to perform row and column operations. Consider $x = (n_1, \ldots, n_d)$ and let $i_1, \ldots, i_d$ be the seats of $\delta^1(x), \ldots, \delta^d(x)$. Look at the i_1-th row. Because $\delta^1(x) = (i_1, n_{i_1})$ is the lowest among all $\delta^j(x)$, we observe from (6.2) that the i_1-th row only has the diagonal entry. We also observe from (6.2) that every entry of a column is divisible by the diagonal entry of this column. Therefore, we may use the i_1-th row to eliminate the nondiagonal entries of the i_1-th column through row operations, without causing other changes.

We continue the process inductively. Having cleared the nondiagonal entries of the i_1-th, $\ldots$, i_{k-1}-th columns, we look at the i_k-th row. Because $\delta_k(x)$ is the lowest among $\delta_j(x)$, $j \geq k$, the entries $a_{i_k, i_j}(T)$ must be zero for $j > k$ for similar reasons. For the entries $a_{i_k, i_j}(T)$ for $j < k$, they are already cleared in the previous steps of the process. Hence the i_k-th row only has the diagonal entry, and we may proceed similarly.

After finishing the process at $k = d$, we obtain the Smith normal form $\mathrm{diag}(T^{n_1}, \ldots, T^{n_d})$ and we are done. $\qquad\square$

Remark 6.2. Both Proposition 2.1 and Proposition 6.1 hold if we replace $\mathbb{F}_q$ by any field. We did not state them in the this generality to avoid distraction.

In [25], Petrogradsky computed a refinement of the Solomon zeta function that not only remembers the index of a sublattice M of $\mathbb{Z}^d$, but also the $\mathbb{Z}$-module structure of $\mathbb{Z}^d/M$. The essence of Petrogradsky's theorem is a local result that holds for all discrete valuation ring.

Theorem 6.3 (Petrogradsky [25, Theorem 3.1(5), Theorem 8.1(2)])**.** *Let R be a discrete valuation ring with uniformizer ϖ and residue field $\mathbb{F}_q$. For a finite-index submodule M of R^d, we define $\lambda_1(M), \ldots, \lambda_d(M)$ by*

$$\frac{R^d}{M} \cong \frac{R}{\varpi^{\lambda_1(M)}} \oplus \cdots \oplus \frac{R}{\varpi^{\lambda_d(M)}}, \quad \lambda_1(M) \geq \cdots \geq \lambda_d(M). \tag{6.5}$$

Consider the series

$$Z(R^d; t_1, \ldots, t_d) := \sum_{(R^d:M)<\infty} t_1^{\lambda_1(M)} \ldots t_d^{\lambda_d(M)} \tag{6.6}$$

in variables $t_1, \ldots, t_d$.

Then $Z(R^d; t_1, \ldots, t_d)$ is a rational function in $q, t_1, \ldots, t_d$ that depends only on d, with denominator dividing $\prod_{j=1}^d (1 - z_j)$, where $z_j = q^{j(d-j)} t_1 t_2 \ldots t_j$.

We note that $Z(R^d; t, t, \ldots, t)$ recovers the Solomon zeta function for R^d. The rational formula for $Z(R^d; t_1, \ldots, t_d)$ is also given in [25], restated below. For a permutation $\pi \in S_d$ of $[d]$, define the set of descents as $D(\pi) = \{i \in [d-1] : \pi(i) > \pi(i+1)\}$, and let $\mathrm{inv}(\pi) := \{i < j : \pi(i) > \pi(j)\}$ denote the number of inversions of π. Define

$$w_D(q) := \sum_{\pi \in S_d, D(\pi)=D} q^{\mathrm{inv}(\pi)}. \tag{6.7}$$

Then

$$Z(R^d; t_1, \ldots, t_d) = \frac{\sum_{D \in [d-1]} \left(w_D(q^{-1}) \prod_{j \in D} z_j \right)}{\prod_{j=1}^d (1 - z_j)}, \tag{6.8}$$

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where $z_j = q^{j(d-j)} t_1 t_2 \ldots t_j$.

Proposition 6.1 immediately expresses the Petrogradsky zeta function $Z(R^d; t_1, \ldots, t_d)$ as a generating function for the W statistics.

Corollary 6.4. *We have*

$$\sum_{x \in X} q^{W(x)} t_1^{n(\delta^d(x))} t_2^{n(\delta^{d-1}(x))} \ldots t_d^{n(\delta^1(x))} = Z(\mathbb{F}_q[[T]]^d; t_1, \ldots, t_d). \tag{6.9}$$

Note that $n(\delta^d(x)), \ldots, n(\delta^1(x))$ is nothing but sorting $n_1, \ldots, n_d$ in descending order, similar to the role of $\lambda_1(M), \ldots, \lambda_d(M)$. When specialized at $t_1 = \cdots = t_d = t$, the left-hand side is equal to $\sum_{x \in X} q^{W(x)} t^{n(x)}$, the subject of Theorem 1.1. Notice that in this special case, the denominator given in Theorem 6.3 is not tight: the denominator is $\prod_{j=1}^d \left(1 - \left(q^{d-j} t\right)^j\right)$, which is divisible by but not equal to the denominator $\prod_{j=1}^d (1 - q^{d-j} t)$ in Theorem 1.1.

Our next goal is to compute the left-hand side of (6.9). This will recover Petrogradsky's theorem, as another application of the spiral shifting operators. We first give a brief overview of several alternative interpretations of the positive-coefficient polynomial $w_D(q)$ given in [25, §5].

6.1 Descents and multiset permutations

Given $D \subseteq [d-1]$ (recall that we shall view it as a set of descents of some permutation), we introduce some notation. Let $D = \{d_1 < \cdots < d_k\}$, where $k = |D|$. For $j = 0, 1, \ldots, k$, write $m_j = d_{j+1} - d_j$, where $d_0 := 0$ and $d_{k+1} := d$.

Cosider a standard inv-preserving bijection frequently used in Schubert calculus:

$$\beta_D : \left\{\pi \in S_d : D(\pi) \subseteq D\right\} \to \left\{\sigma \in \mathrm{Map}([d], \{0, \ldots, k\}) : |\sigma^{-1}(i)| = m_i\right\}$$
$$\pi \mapsto (\pi(i) \mapsto \text{unique } j \text{ such that } d_j < i \leq d_{j+1}). \tag{6.10}$$

For example, if $d = 5$, $D = \{2, 4\}$, $\pi = 25|13|4$ (the verticle bars denote positions where descents are allowed; note how they encode $(m_0, m_1, m_2) = (2, 2, 1)$), then $\beta_D(\pi)$ sends $2, 5$ to 0, and $1, 3$ to 1, and 4 to 2. The two-line notation $\begin{pmatrix} 25 & 13 & 4 \\ 00 & 11 & 2 \end{pmatrix}$ is instructive. We shall view $\beta(\pi)$ as a permutation of the multiset $0^{m_0} 1^{m_1} \ldots k^{m_k}$; in this example, $\beta_D(\pi)$ is the multiset permutation 10120. It is instructive to read $\beta_D(\pi)$ by permutating the columns of the two-line notation above, such that the first row is sorted: $\begin{pmatrix} 12345 \\ 10120 \end{pmatrix}$. We always have $\mathrm{inv}(\pi) = \mathrm{inv}(\beta_D(\pi))$. In this example, both numbers of inversions are 4.

The map β_D induces a bijection

$$\left\{\pi \in S_d : D(\pi) = D\right\} \xrightarrow{\cong} \left\{\sigma : \text{For any } j \in [d], \text{ there is } i_1 < i_2, \sigma(i_1) = j, \sigma(i_2) = j-1\right\}. \tag{6.11}$$

We call the latter the set of **strict** multiset permutations, namely, multiset permutations that has at least a j put before a $j-1$ for any $1 \leq j \leq d$. For example, 10120 is not a strict multiset permutation because every 1 is before every 2, and this corresponds to the fact that the second vertible bar of $\pi = 25|13|4$ is not actually a descent.

The bijection (6.11) gives the alternative formula [25, Theorem 5.3(1)]:

$$w_D(q) = \sum_{\sigma \text{ strict}} q^{\text{inv}(\sigma)}, \tag{6.12}$$

where the sum is over all strict multiset permutations of $0^{m_0} 1^{m_1} \ldots k^{m_k}$.

Another formula for $w_D(q)$ can be obtained from a formula of MacMahon [1, §3.4]

$$\sum_{\pi \in S_d, D(\pi) \subseteq D} q^{\text{inv}(\pi)} = \begin{bmatrix} d \\ m_0, m_1, \ldots, m_k \end{bmatrix}_q, \tag{6.13}$$

which can be viewed as the Schubert cell decomposition of the flag variety $\text{Fl}(D; d) := \text{Fl}(d_1, \ldots, d_k; d)$. We denote by $\begin{bmatrix} d \\ D \end{bmatrix}_q$ the Gaussian q-multinomial polynomial $\begin{bmatrix} d \\ m_0, m_1, \ldots, m_k \end{bmatrix}_q$. Using the Möbius inversion formula and the expression of the Möbius function on the Boolean poset of subsets of $[d-1]$ (see for example, [24, Proposition 2.44]), one obtains

$$w_D(q) = \sum_{D' \subseteq D} (-1)^{|D| - |D'|} \begin{bmatrix} d \\ D' \end{bmatrix}_q. \tag{6.14}$$

Petrogradsky [25, §3] used this instead of (6.7) as the original definition of $w_D(\lambda)$.

6.2 Alternative proof of Petrogradsky's theorem for $\mathbb{F}_q[[T]]^d$

We now compute the left-hand side of (6.9). Let Γ' be the subsemigroup of Γ (freely) generated by $g_1^d, g_2^{d-1}, \ldots, g_d^1$. We notice that the effect of g_{d+1-j}^j on X is raising the level of the highest j points of $\Delta(x)$ by 1 while fixing their seats; this is because g_{d+1-j} is spiral shifting within the highest j elements, so shifting j times completes a full cycle. For $x \in X$, we define the **seat permutation** of x to be the permutation $i(\delta^1(x)), i(\delta^2(x)), \ldots, i(\delta^d(x))$.

Lemma 6.5. *There is an orbit decomposition of X:*

$$X = \bigsqcup_{\pi \in S_d} \Gamma' \cdot x_\pi \tag{6.15}$$

for certain $d!$ elements $x_\pi \in X$. Moreover, each orbit consists precisely of elements of X with seat permutation π.

Proof. The index of Γ' in Γ is $d \cdot (d-1) \cdot \cdots \cdot 1 = d!$, and Γ is clearly a disjoint union of $d!$ cosets of Γ'. Say $\Gamma = \bigsqcup_{k=1}^{d!} a_k \Gamma'$ for some $a_k \in \Gamma$. Then we have a decomposition of X into $d!$ disjoint Γ' orbits:

$$X = \Gamma \cdot \mathbf{0} = \bigsqcup_{k=1}^{d!} \Gamma' \cdot (a_k \cdot \mathbf{0}) \tag{6.16}$$

By the discussion above this lemma, each orbit has a constant seat permutation. Let π_k be the seat permutation of $a_k \cdot \mathbf{0}$ (and thus the seat permutation of every element in the k-th orbit). Every π occurs as some π_k, since there exists an element of any given seat permutation (for instance, if $\pi = i_1 \ldots i_d$, then consider the element x with $\Delta(x) = \{(i_1, 1), \ldots, (i_d, d)\}$). Because there are $d!$ permutations and $d!$ orbits, all π_k must be distinct. Hence the proof of the lemma is complete by relabeling. $\square$

Define
$$\mathrm{Cont}'(x) := q^{W(x)} t_1^{n(\delta^d(x))} t_2^{n(\delta^{d-1}(x))} \dots t_d^{n(\delta^1(x))}, \tag{6.17}$$

so that the left-hand side of (6.9) is $\sum_{x \in X} \mathrm{Cont}'(x)$. We have

$$\mathrm{Cont}'(g^j_{d+1-j}(x)) = q^{W(g^j_{d+1-j}(x))} t_1^{n(\delta^d(x))+1} \dots t_j^{n(\delta^{d+1-j}(x))+1} t_{j+1}^{n(\delta^{d-j}(x))} \dots t_d^{n(\delta^1(x))} \tag{6.18}$$

$$= q^{j(d-j)} t_1 t_2 \dots t_j \, \mathrm{Cont}'(x) \tag{6.19}$$

$$=: z_j \, \mathrm{Cont}'(x), \tag{6.20}$$

where the first equality results from the discussion at the beginning of this section, and the second equality is by Lemma 5.2. The orbit decomposition thus imply the following formula.

Theorem 6.6. *We have*

$$\sum_{x \in X} q^{W(x)} t_1^{n(\delta^d(x))} t_2^{n(\delta^{d-1}(x))} \dots t_d^{n(\delta^1(x))} = \frac{1}{(1-z_1)\dots(1-z_d)} \sum_{\pi \in S_d} \mathrm{Cont}'(x_\pi). \tag{6.21}$$

Proof. Perform the same proof as Theorem 5.4 for each orbit in Lemma 6.5. $\qquad\square$

In particular, we have proved that $Z(\mathbb{F}_q[[T]]^d; t_1, \dots, t_d)$ is rational with the desired denominator. Moreover, we obtain yet another formula $\sum_{\pi \in S_d} \mathrm{Cont}'(x_\pi)$ for the numerator, which implies that the numerator is a positive-coefficient polynomial in $q, t_1, \dots, t_d$ with total coefficient $d!$ (i.e., evaluates to $d!$ at $q = t_1 = \dots = t_d = 1$).

We now verify that (6.21) does recover Petrogradsky's formula (6.8). This means that we shall prove

$$\text{Claim:} \quad \sum_{\pi \in S_d} \mathrm{Cont}'(x_\pi) = \sum_{D \in [d-1]} \left(w_D(q^{-1}) \prod_{j \in D} z_j \right). \tag{6.22}$$

We first observe that x_π is given by the strict multiset permutation $\beta_{D(\pi)}(\pi)$. For example, if $d = 5$ and $\pi = 25|14|3$ (labeling all descents with verticle bars), then $\beta_{D(\pi)}(\pi) = 10210$ and $x_\pi = (1,0,2,1,0)$. Indeed, from the orbit decomposition in Lemma 6.5, the element x_π must be the "lowest possible" element with seat permutation π. If we write $\pi = i_1 i_2 \dots i_d$ and $\Delta(x_\pi) = \{(i_1, n_{i_1}) \prec (i_2, n_{i_2}) \prec \dots \prec (i_d, n_{i_d})\}$, then x_π is obtained from greedily assigning the lowest possible values for $n_{i_1}, n_{i_2}, \dots, n_{i_d}$ such that $(i_1, n_{i_1}) \prec (i_2, n_{i_2}) \prec \dots \prec (i_d, n_{i_d})$. The resulting x_π is precisely $\beta_{D(\pi)}(\pi)$, because we start at $n_{i_1} = 0$ and the level of the next point must rise by 1 if the seat descends.

We then claim that

$$t_1^{n(\delta^d(x_\pi))} t_2^{n(\delta^{d-1}(x_\pi))} \dots t_d^{n(\delta^1(x_\pi))} = \prod_{j \in D} y_{d-j}, \tag{6.23}$$

where $y_j := t_1 t_2 \dots t_j$. The proof is clearer if illustrated on an example. Consider $\pi = 16|235|48|7$, then $D(\pi) = \{2, 5, 7\}$ and $(m_0, \dots, m_3) = (2, 3, 2, 1)$. We have $\Delta(x_\pi) = \begin{pmatrix} 16 & 235 & 48 & 7 \\ 00 & 111 & 22 & 3 \end{pmatrix}$, which reads $\{(1,0), (6,0), \dots, (7,3)\}$. Then reading the levels of $\Delta(x_\pi)$ from high to low, we obtain

$$t_1^{n(\delta^d(x_\pi))} t_2^{n(\delta^{d-1}(x_\pi))} \dots t_d^{n(\delta^1(x_\pi))} = t_1^3 t_2^2 t_3^2 t_4^1 t_5^1 t_6^1 t_7^0 t_8^0 = y_1 y_3 y_6 = \prod_{j \in D} y_{d-j}. \tag{6.24}$$

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We now compute $W(x_\pi)$. Write $\pi = i_1 i_2 \ldots i_d$, $D(\pi) = \{d_1 < \cdots < d_k\}$, and

$$\Delta(x_\pi) = \begin{pmatrix} i_1 & \cdots & i_{d_1} & i_{d_1+1} & \cdots & i_{d_2} & \cdots & i_{d_k+1} & \cdots & i_d \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & k & \cdots & k \end{pmatrix} \tag{6.25}$$

Note that the sequence $n(\delta^1(x_\pi)), \ldots, n(\delta^d(x_\pi))$ is the second row of (6.25), which only depends on $D = D(\pi)$. We denote the sequence by $n_1(D), \ldots, n_d(D)$; it is m_0 copies of 0's, followed by m_1 copies of 1's, ..., m_k copies of k's. Then for $a < b$, the columns $\delta^a(x_\pi), \delta^b(x_\pi)$ have distance $n_b(D) - n_a(D)$ if $i_a < i_b$, or the above number minus one if $i_a > i_b$. Hence

$$W(x_\pi) = -\mathrm{inv}(\pi) + \sum_{1 \le a < b \le d} (n_b(D) - n_a(D)). \tag{6.26}$$

We shall simplify $\sum_{a<b}(n_b(D) - n_a(D))$. For $0 \le r < s \le k$, the difference $s - r$ occurs as the difference $n_b(D) - n_a(D)$ for $m_r m_s$ times. Thus

$$\sum_{a<b}(n_b(D) - n_a(D)) = \sum_{0 \le r < s \le k} m_r m_s (s - r). \tag{6.27}$$

One may verify that

$$\sum_{0 \le r < s \le k} m_r m_s (s-r) = m_0(m_1+\cdots+m_k)+(m_0+m_1)(m_2+\cdots+m_k)+\cdots+(m_0+\cdots+m_{k-1})m_k$$
$$\tag{6.28}$$

by noting that the term $m_r m_s$ occurs $s - r$ times on the right-hand side. Recalling that $m_0, \ldots, m_k$ are the gaps between consecutive terms of $0, d_1, \ldots, d_k, d$, we get

$$\sum_{a<b}(n_b(D) - n_a(D)) = \sum_{j \in D} j(d - j). \tag{6.29}$$

Putting the above together, we get

$$\mathrm{Cont}'(x_\pi) = q^{-\mathrm{inv}(\pi)} q^{\sum_{j \in D(\pi)} j(d-j)} \prod_{j \in D(\pi)} y_{d-j} = q^{-\mathrm{inv}(\pi)} \prod_{j \in D(\pi)} z_{d-j}. \tag{6.30}$$

In particular, we obtain a formula for the numerator of Petrogradsky's zeta function as a summation over all partitions of S_d that keeps track of the number of inversions and the set of descents:

$$\sum_{x \in X} q^{W(x)} t_1^{n(\delta^d(x))} t_2^{n(\delta^{d-1}(x))} \cdots t_d^{n(\delta^1(x))} = \frac{1}{(1-z_1)\ldots(1-z_d)} \sum_{\pi \in S_d} \left(q^{-\mathrm{inv}(\pi)} \prod_{j \in D(\pi)} z_{d-j} \right). \tag{6.31}$$

Collecting all π with the same $D(\pi)$, we get

$$\sum_{x \in X} q^{W(x)} t_1^{n(\delta^d(x))} t_2^{n(\delta^{d-1}(x))} \cdots t_d^{n(\delta^1(x))} = \frac{1}{(1-z_1)\ldots(1-z_d)} \sum_{D \subseteq [d-1]} \left(w_D(q^{-1}) \prod_{j \in D} z_{d-j} \right). \tag{6.32}$$

The slight difference between the above formula and (6.8) can be addressed by considering the involution on the power set of $[d-1]$ defined by $D \mapsto D^t := \{d - j : j \in D\}$, and the

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involution on S_d that sends $\pi = i_1 \ldots i_d$ to the "retrograde inversion" $\pi^t := (d+1-i_d)(d+1-i_{d-1})\ldots(d+1-i_1)$.[3] We have $\mathrm{inv}(\pi^t) = \mathrm{inv}(\pi)$ and $D(\pi^t) = D(\pi)^t$, so that $w_D(q) = w_{D^t}(q)$. By a change of variable $D \to D^t$, the formula (6.32) implies (6.8), and the alternative proof of Petrogradsky's theorem is complete. As a takeaway of the above analysis, we rewrite (6.31) in a slightly better form, obtained from (6.31) with $\pi \mapsto \pi^t$, and state some functional equations as direct consequences of this form.

Corollary 6.7 ([25]). *We have*

$$Z(\mathbb{F}_q[[T]]^d; t_1, \ldots, t_d) = \frac{\sum_{\pi \in S_d}\left(q^{-\mathrm{inv}(\pi)} \prod_{j \in D(\pi)} z_j\right)}{\prod_{j=1}^d (1 - z_j)}, \tag{6.33}$$

where $z_j = q^{j(d-j)} t_1 t_2 \ldots t_j$. *Moreover, the numerator is a polynomial in* $q^{-1}, z_1, \ldots, z_{d-1}$ *that is invariant under the simultaneous change of variable* $z_j \mapsto z_{d-j}$. *In particular,* $Z(\mathbb{F}_q[[T]]^d; t_1, \ldots, t_d)$ *is a rational function* $Z(q; z_1, \ldots, z_d)$ *in* $q, z_1, \ldots, z_d$ *that satisfies functional equations*

(a) $Z(q; z_{d-1}, \ldots, z_1, z_d) = Z(q; z_1, \ldots, z_{d-1}, z_d)$;

(b) $Z(q^{-1}; z^{-1}, \ldots, z_d^{-1}) = (-1)^d q^{\binom{d}{2}} Z(q; z_1, \ldots, z_d)$, *or equivalently,*

$$Z(\mathbb{F}_q[[T]]^d; t_1, \ldots, t_d)|_{q \mapsto q^{-1}, t_i \mapsto t_i^{-1}} = (-1)^d q^{\binom{d}{2}} Z(\mathbb{F}_q[[T]]^d; t_1, \ldots, t_d). \tag{6.34}$$

Proof. The assertion (a) is obvious from the previous discussions. For the assertion (b), notice that the permutation $\pi' = w_0 \pi$ where $w_0 = d(d-1)\ldots 1$ is the longest word satisfies $\mathrm{inv}(\pi') = \binom{d}{2} - \mathrm{inv}(\pi)$ and $D(\pi') = [d-1] \setminus D(\pi)$. Thus

$$\begin{aligned}
Z(q^{-1}; z_1^{-1}, \ldots, z_d^{-1}) &= \frac{\sum_{\pi \in S_d}\left(q^{\mathrm{inv}(\pi')} \prod_{j \in D(\pi')} z_j^{-1}\right)}{\prod_{j=1}^d (1 - z_j^{-1})} \\
&= \frac{(-1)^d z_1 \ldots z_d}{\prod_{j=1}^d (1 - z_j)} \cdot \sum_{\pi \in S_d}\left(q^{\binom{d}{2} - \mathrm{inv}(\pi)} \cdot z_1^{-1} \ldots z_d^{-1} \prod_{j \in D(\pi)} z_j\right) \\
&= (-1)^d q^{\binom{d}{2}} Z(q; z_1, \ldots, z_d).
\end{aligned} \tag{6.35}$$

The final equivalent statement is because as $q, t_1, \ldots, t_j \mapsto q^{-1}, t_1^{-1}, \ldots, t_d^{-1}$, we have $z_j \mapsto z_j^{-1}$. $\qquad\square$

Remark 6.8. As a final remark, we compare our proof to other known proofs of (6.33). Petrogradsky's proof builds upon results that for each abelian p-group G, count subgroups $H \subseteq G$ of a given cotype (i.e., the isomorphism class). His proof arrives at a formula in terms of Gaussian q-multinomial polynomials ((6.8) but with $w_D(q)$ defined by (6.14)), which turns out somewhat miraculously to be interpretable as a permutation sum, and hence the numerator of (6.8) has nonnegative coefficients. The work of du Sautoy and Lubotzky [9, Theorem 5.9] recovers (6.33)

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[3] "Retrograde inversion" is a musical term that describes such a transformation on a melody. Equivalently, $\pi^t = w_0 \pi w_0$, where $w_0 = d(d-1)\ldots 1$ is the longest word of S_d.

as a special case where $G = \mathrm{GL}_d$ and ρ is the standard representation. Their proof uses p-adic integral over $\mathrm{GL}_d(\mathbb{Z}_p)$ and the Iwahori decomposition, and reaches the permutation sum (or more generally, a sum over the (affine) Weyl group) and the functional equation (6.34). See also the concluding remark of [7, §2]. Our proof is based on a matrix normal form (6.1), and it leads to the permutation sum of (6.33) via the orbit decomposition in Lemma 6.5.

Acknowledgements

References

[1] G. E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[2] M. Bagnarol, B. Fantechi, and F. Perroni. On the motive of Quot schemes of zero-dimensional quotients on a curve. *New York J. Math.*, 26:138–148, 2020.

[3] T. Becker. Stability and Buchberger criterion for standard bases in power series rings. *J. Pure Appl. Algebra*, 66(3):219–227, 1990.

[4] D. Bejleri, D. Ranganathan, and R. Vakil. Motivic Hilbert zeta functions of curves are rational. *J. Inst. Math. Jussieu*, 19(3):947–964, 2020.

[5] E. Bifet. Sur les points fixes du schéma $\mathrm{Quot}_{\mathscr{O}_X^r/X/k}$ sous l'action du tore $\mathbf{G}_{m,k}^r$. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(9):609–612, 1989.

[6] J. Bryan and A. Morrison. Motivic classes of commuting varieties via power structures. *J. Algebraic Geom.*, 24(1):183–199, 2015.

[7] G. Chinta, N. Kaplan, and S. Koplewitz. The cotype zeta function of $\mathbb{F}^d$. Preprint. Available at arXiv:1708.08547.

[8] H. Cohen and H. W. Lenstra, Jr. Heuristics on class groups of number fields. In *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, volume 1068 of *Lecture Notes in Math.*, pages 33–62. Springer, Berlin, 1984.

[9] M. P. F. du Sautoy and A. Lubotzky. Functional equations and uniformity for local zeta functions of nilpotent groups. *Amer. J. Math.*, 118(1):39–90, 1996.

[10] W. Feit and N. J. Fine. Pairs of commuting matrices over a finite field. *Duke Math. J.*, 27:91–94, 1960.

[11] L. Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. *Math. Ann.*, 286(1-3):193–207, 1990.

[12] L. Göttsche and V. Shende. Refined curve counting on complex surfaces. *Geom. Topol.*, 18(4):2245–2307, 2014.

[13] A. Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[14] A. A. Henni and D. M. Guimarães. A note on the ADHM description of Quot schemes of points on affine spaces. *Internat. J. Math.*, 32(6):Paper No. 2150031, 17, 2021.

[15] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math.*, 79:109–326, 1964.

[16] Y. Huang. Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity. Submitted. Available at arXiv: 2110.15566, 2021.

[17] Y. Huang. Counting on the variety of modules over the quantum plane. *Algebr. Comb.*, to appear, 2022. Available at arXiv: 2110.15570.

[18] R. Ito. Every semilinear set is a finite union of disjoint linear sets. *J. Comput. System Sci.*, 3:221–231, 1969.

[19] N. Kaplan, J. Marcinek, and R. Takloo-Bighash. Distribution of orders in number fields. *Res. Math. Sci.*, 2:Art. 6, 57, 2015.

[20] A. Lubotzky and D. Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

[21] D. Maulik and Z. Yun. Macdonald formula for curves with planar singularities. *J. Reine Angew. Math.*, 694:27–48, 2014.

[22] S. Monavari and A. T. Ricolfi. On the motive of the nested Quot scheme of points on a curve. *J. Algebra*, 610, 2022.

[23] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[24] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[25] V. M. Petrogradsky. Multiple zeta functions and asymptotic structure of free abelian groups of finite rank. *J. Pure Appl. Algebra*, 208(3):1137–1158, 2007.

[26] L. Solomon. Zeta functions and integral representation theory. *Advances in Math.*, 26(3):306–326, 1977.