Cohen–Lenstra series from module statistics and matrix enumeration

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Main motivation

Formulate a common framework (the Cohen–Lenstra series) to

- unify existing matrix enumeration results of distinct flavors.
- motivate radically new matrix enumeration problems.

Selected classical results

$$|\mathrm{Nilp}_n(\mathbb{F}_q)| = q^{n^2-n}$$ (Fine–Herstein '58 on nilpotent matrices)

$$\sum_{n\geq 0} \frac{|\{A,B\in \mathrm{Mat}_n(\mathbb{F}_q): AB=BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|}x^n = \prod_{i,j\geq 1}\frac{1}{1-x^iq^{2-j}}$$

(Feit–Fine '60)

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(Fulman–Guralnick '18)

Key definition

To put the above results in a common framework, define for a commutative algebra R over $\mathbb{F}_q$ the **Cohen–Lenstra series** by

$$\widehat{Z}_R(x) = \widehat{Z}_{R/\mathbb{F}_q}(x) := \sum_{M/R}\frac{1}{|\mathrm{Aut}\, M|}x^{\dim_{\mathbb{F}_q} M},$$

where

- M ranges over all isomorphism classes of R-modules that are finite-dimensional as $\mathbb{F}_q$-vector spaces.
- $|\mathrm{Aut}\, M|$ is the size of the automorphism group of M.
- $\dim_{\mathbb{F}_q} M$ is the dimension of M as an $\mathbb{F}_q$-vector space.

In some sense, $\widehat{Z}_R(x)$ is about the statistics of finite-dimensional R-modules distributed under the $1/|\mathrm{Aut}\, M|$ measure (originally studied by Cohen and Lenstra for Dedekind domain R for number-theoretical purposes).

Classical results translated

- Fine–Herstein $\iff$ $\widehat{Z}_{\mathbb{F}_q[[X]]}(x) = \prod_{i\geq 1} 1/(1-xq^{-i})$.
- Feit–Fine $\iff$ $\widehat{Z}_{\mathbb{F}_q[X,Y]}(x) = \prod_{i,j\geq 1} 1/(1-x^iq^{2-j})$.
- Fulman–Guralnick $\iff$ $\widehat{Z}_{\mathbb{F}_q[[X,Y]]}(x) = \prod_{i,j\geq 1} 1/(1-x^iq^{-j})$.

Translation recipe

If $R=\mathbb{F}_q[X_1,\dots,X_m]/(f_1,\dots,f_r)$, then

$$\widehat{Z}_R(x) = \sum_{n\geq 0}\frac{|K_n(R)|}{|\mathrm{GL}_n(\mathbb{F}_q)|}x^n,$$

where $K_n(R)$ is the set of all tuples $(A_1,\dots,A_m)$ such that

- $A_1,\dots,A_m$ are commuting matrices in $\mathrm{Mat}_n(\mathbb{F}_q)$.
- For $1\leq k\leq r$, the matrix $f_k(A_1,\dots,A_k)$ (which makes sense!) is zero.

If $R=\mathbb{F}_q[[X_1,\dots,X_m]]/(f_1,\dots,f_r)$, then the same formula holds, except in the definition of $K_n(R)$, the matrices A_i are in addition required to be nilpotent.

What's known about $\widehat{Z}_R(x)$

The notion of $\widehat{Z}_R(x)$ allows to tell new problems from old easily. The behavior of $\widehat{Z}_R(x)$ is determined (very sensitively!) by the local geometry of the affine scheme $\mathrm{Spec}\, R$.

Locality.

$$\widehat{Z}_R(x) = \prod_p \widehat{Z}_{\widehat{R}_p}(x),$$

where $\widehat{R}_p$ ranges over the completed localizations of R at all maximal ideals p.

Known cases.

(a) $R=\mathbb{F}_q[[X]]$, $\widehat{Z}_R(x)=\prod_{i\geq 1}1/(1-xq^{-i})$. (Fine–Herstein)

(b) $R=\mathbb{F}_q[[X,Y]]$, $\widehat{Z}_R(x)=\prod_{i,j\geq 1}1/(1-x^iq^{-j})$. (Fulman–Guralnick)

(c) $X:=\mathrm{Spec}\, R$ is a smooth curve, then $\widehat{Z}_R(x)=\prod_{i\geq 1}Z_X(xq^{-i})$.

(d) $X:=\mathrm{Spec}\, R$ is a smooth surface, then $\widehat{Z}_R(x)=\prod_{i,j\geq 1}Z_X(x^iq^{-j})$.

Here, $Z_X(t)$ denotes the Hasse–Weil zeta function of X, e.g., $Z_{\mathbb{A}^1}(t)=1/(1-qt)$. Formulas (c)(d) follow naturally from (a)(b) + locality of $\widehat{Z}_R(x)$.

The case for other R is radically different and there is almost nothing we can say, except for the main theorems below.

Contributions

Using notation

$$(a;t)_n := (1-a)(1-at)\dots(1-at^{n-1}),$$
$$(a;t)_\infty := (1-a)(1-at)(1-at^2)\dots.$$

Theorem 1: a singular result (H. '21, arXiv:2110.15566)

The count of pairs of mutually annihilating matrices is given by the formula

$$\sum_{n=0}^{\infty}\frac{|\{A,B\in\mathrm{Mat}_n(\mathbb{F}_q): AB=BA=0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|}x^n = \frac{1}{(x;q^{-1})^2_\infty}H_q(x),$$

where

$$H_q(x) := \sum_{k=0}^{\infty}\frac{q^{-k^2}x^{2k}}{(q^{-1};q^{-1})_k}(xq^{-k-1};q^{-1})_\infty.$$

Moreover, $H_q(x)$ is an entire function in $x\in\mathbb{C}$.

Remarks

- The LHS is $\widehat{Z}_R(x)$ where $R=\mathbb{F}_q[X,Y]/(XY)$.
- $\mathrm{Spec}\, R$ is two lines intersecting at a nodal singularity.
- Theorem 1 is interesting even without the explicit formula of $H_q(x)$: the point is that $\widehat{Z}_R(x)$ admits such a factorization for some entire function $H_q(x)$.
- $H_q(x)$ can be viewed as a local invariant attached to the nodal singularity, though the relation to its geometry is unknown.

Methods

- The counting is direct, using elementary linear algebra.
- The factorization requires standard techniques manipulating Young diagrams, especially the Durfee squares.
- It is coincidental that elementary methods work! It is far from the case for other singularities. A geometric explanation is lacking.

Theorem 2: a noncommutative result (H. '21, arXiv:2110.15570)

Let ζ be a primitive m-th root of unity of $\mathbb{F}_q$ (so $\zeta^m=1$). Then

$$\sum_{n=0}^{\infty}\frac{|\{A,B\in\mathrm{Mat}_n(\mathbb{F}_q): AB=\zeta BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|}x^n = \prod_{i=1}^{\infty}F_m(x^i;q),$$

where

$$F_m(x;q) := \frac{1-x^m}{(1-x)(1-x^mq)}\cdot\frac{1}{(x;q^{-1})_\infty}.$$

Remarks

- Recovers Feit–Fine '60 as the special case $\zeta=1$, by substituting $m=1$ in the formula.
- The LHS can be viewed as a noncommutative version of $\widehat{Z}_R(x)$ where $R=\mathbb{F}_q\{X,Y\}/(XY-\zeta YX)$, the **quantum plane**.
- The first factor of $F_m(x;q)$ can be viewed (in some precise sense) as the contribution of pairs with invertible A, and the second factor the contribution of pairs with nilpotent A.

Methods

- The counting of pairs with nilpotent A is explicit entry-wise equation solving.
- The counting with invertible A boils down to classifying matrices B such that B is similar to ζB. There is a nice classification in terms of the Smith normal form.

Conjecture: other singularities

Let $\mathrm{Spec}\, R$ be a singular curve, and $\mathrm{Spec}\, \widetilde{R}$ be its normalization (i.e., resolution of singularity). Then $\widehat{Z}_R(x)$ has a factorization

$$\widehat{Z}_R(x) = \widehat{Z}_{\widetilde{R}}(x)H_X(x)$$

such that $H_X(x)$ is entire.

Remarks

- The conjecture predicts that $\widehat{Z}_R(x)$ has a "natural" factorization with one of the factors depending on the normalization of R. (For now, "natural" = the other factor is entire.)
- The conjecture is local and only depends on the singularities of X.
- The conjecture is verified for the nodal singularity (Theorem 1). We have obtained positive evidences in the case of a cusp (joint work in progress).