Punctual Quot scheme on cusp via Gröbner stratification

Yifeng Huang

University of British Columbia

joint with Ruofan Jiang

Theorem (HJ23, arxiv: 2305.06411)

Let $X = \{x^2 = y^3\}$ *be the affine cusp curve. Then*

$$\sum_{n=0}^{\infty} [\mathrm{Quot}_{d,n}(X)] t^n := \sum_{n=0}^{\infty} [\{\mathcal{O}_X^d \twoheadrightarrow M : \dim \operatorname{supp} M = 0, \deg M = n\}] t^n$$

is rational in t *with denominator* $(1 - \mathbb{L}t)(1 - \mathbb{L}^2 t) \ldots (1 - \mathbb{L}^d t)$.

Background – Definitions

Let $k = \mathbb{C}$ or $\mathbb{F}_q$, and X be a k-variety.

- **Motive**: The motive $[X]$ of an variety X is the equivalence class of X in $K_0(\mathrm{Var}_k)$ (the Grothendieck ring of varieties) where we identify $[V] = [V \smallsetminus Z] + [Z]$ for closed subvariety $Z \subseteq V$ and $[V \times W] = [V] \cdot [W]$. This is a geometric version of "point counting over $\mathbb{F}_q$" but works for $\mathbb{C}$-varieties as well. Write $\mathbb{L} = [\mathbb{A}^1]$.
- **Quot scheme**: $\mathrm{Quot}_{d,n}(X)$ is the moduli space parametrizing quotient sheaves $\mathcal{O}_X^d \twoheadrightarrow M$ of zero-dimensional support and length n.
- **Commuting variety**: If X is affine, $C_n(X)$ parametrizes tuples of commuting matrices satisfying defining equations of X.
- **Stack of coherent sheaves**: $\mathrm{Coh}_n(X)$ is the moduli stack parametrizing coherent sheaves of zero-dimensional support and length n.
- For X affine, $\mathrm{Coh}_n(X)$ is the stack quotient $[C_n(X)/\,\mathrm{GL}_n]$. The motive of $\mathrm{Coh}_n(X)$ satisfies $[\mathrm{Coh}_n(X)] = [C_n(X)]/[\mathrm{GL}_n]$.

Background – Old and new results

X	$[\mathrm{Coh}_n(X)]$	$[\mathrm{Quot}_{1,n}(X)]$	$[\mathrm{Quot}_{d,n}(X)]$
smooth curve	[FH58*|CL83*]	[well-known]	[Sol77*|Bif89, etc.]
smooth surface	[FF60*|BM15]	[ES87|Göt01]	(χ) [OP21]
singular curve	(eg) [H.23]	[GS14, BRV20]	(eg) [HJ23]
singular surface	open	(eg, χ) [GNS17]	open
essentially...	matrix count	ideal count	submodule count

[FH58]: Fine and Herstein; count nilpotent matrices
[CL83]: Cohen and Lenstra
[well-known]: ideals of $k[[T]]$ are (T^n); | $\mathrm{Quot}_{1,n}(X) = \mathrm{Sym}^n(X)$.
[Sol77]: Solomon; lattice zeta function
[Bif89, etc.]: Bifet; generalizations by BFP20, MR22
[FF60]: Feit and Fine; count matrices $AB = BA$
[BM15]: Bryan and Morrison
[ES87]: Ellingsrud and Strømme; $\mathrm{Hilb}_n(\mathbb{C}^2)$
[Göt01]: Göttsche
[OP21]: Oprea and Pandharipande
[H.23]: H.; node case; count matrices $AB = BA = 0$
[GS14]: Göttsche and Shende
[BRV20]: Bejleri, Ranganathan and Vakil
[GNS17]: Gyenge, Némenthi and Szendrői

*: not in motivic language
[local|global]: local and global results
(eg): known only for some (families of) examples
(χ): known in terms of related invariants such as (virtual) Euler characteristics, but not motive

Singular curves

Let $Q_{d,X}(t) = \sum_{n=0}^{n}[\mathrm{Quot}_{d,n}(X)]t^n$ and $\widehat{Z}_X(t) = \sum_{n=0}^{n}[\mathrm{Coh}_n(X)]t^n$.
Then $Q_{1,X}(t)$ satisfies

- **Rationality**: $Q_{1,X}(t)$ is rational with denominator dictated by the normalization of X (equivalently, numbers of branches around singularities). The numerator depends on the singularities only. [BRV20]
- **Functional Equation**: Under mild assumptions on the singularities, $Q_{1,X}(t)$ satisfies a functional equation $t \mapsto 1/(\mathbb{L}t)$, which means the numerator has a certain symmetry. [GS14]

In [H. 23], I discovered a **"rationality"** statement for $\widehat{Z}_X(t)$ if X has only nodal singularities. Nothing was known for $Q_{d,X}(t)$ if X is singular and $d \geq 2$.

Goal

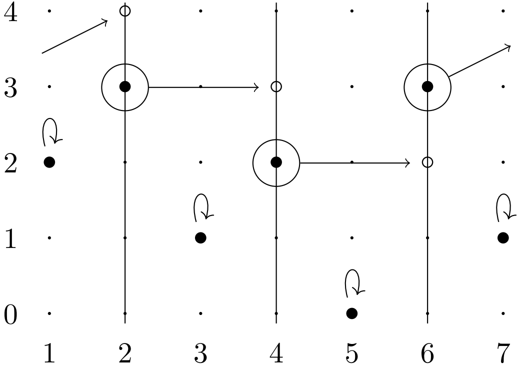
Investigate analogous phenomena for $Q_{d,X}(t)$ and $\widehat{Z}_X(t)$ for curve singularities in general.

Algebraic ingredient: Gröbner stratification

Let $X = \{x^2 = y^3\}$ be the affine cusp curve. Now prove [HJ23].

- Recall $\mathrm{Quot}_{d,n}(X) = \{\mathcal{O}_X^d \twoheadrightarrow M\}$. Taking the kernel, $\mathrm{Quot}_{d,n}(X) = \{N \subseteq \mathcal{O}_X^d : \mathcal{O}_X^d / K$ has zero-dimensional support and length $n\}$.
- Locally speaking, we are classifying submodules of R^d (where $R = k[[T^2, T^3]]$) of codimension n as a k-vector subspace.
- The theory of reduced Gröbner bases for power series ring (aka standard bases) does exactly this job. Here R is only a subring of $k[[T]]$, but the theory extends with minor modification.
- We stratify our moduli space into strata, each indexed by a "leading term datum" that is discrete and combinatorial.
- We try to classify submodules that belong to each stratum. This requires a Buchberger criterion to test if a submodule does have a prescribed leading term datum.
- It turns out that each stratum is parametrized by solutions of a matrix equation; we don't fully understand them, but we observe **eventual stability.**

Combinatorial ingredient: Spiral raising operator



$$\gamma_{(5)}(2,3,1,2,0,3,1) = (2,4,1,3,0,2,1)$$

- $\bullet$ unmoved points
- $\odot$ origins of moved points
- $\circ$ destinations of moved points
- $|$ columns of moved points

- Motives of strata are eventually stable (up to correction) with respect to the "spiral raising operators" performed on the leading term data, essentially d-tuples.
- $\gamma_{(j)} (1 \leq j \leq d)$ fixes the lowest $j-1$ points and moves the rest.
- We show that these operators commute and act freely transitively.
- **Upshot**: Break the sum over all data into finitely many **orbit** sums!

Combining ingredients: Orbit sum

$(J(1),K(0))$ $1\cdot q\cdot q^3t^1$	$\overset{\gamma_{(3)}}{\dashrightarrow}$ $(J(1),K(1))$ $1\cdot q\cdot q^4t^2$	$\dashrightarrow$ $(J(1),K(2))$ $q\cdot q\cdot q^6t^3$	$\dashrightarrow$ $(J(1),K(3))$ $q^2\cdot q\cdot q^8t^4$	$\rightarrow \cdots$
$\gamma_{(2)}\downarrow$ $(K(1),J(1))$ $1\cdot q\cdot q^4t^2$	$(K(2),J(1))$ $q\cdot q\cdot q^5t^3$	$(K(3),J(1))$ $q^2\cdot q\cdot q^7t^4$	$(K(4),J(1))$ $q^2\cdot q\cdot q^9t^5$	$\rightarrow \cdots$
$(J(2),K(1))$ $1\cdot q\cdot q^5t^3$	$(J(2),K(2))$ $q\cdot q\cdot q^6t^4$	$(J(2),K(3))$ $q^2\cdot q\cdot q^8t^5$	$(J(2),K(4))$ $q^2\cdot q\cdot q^{10}t^6$	$\rightarrow \cdots$
$(K(2),J(2))$ $q\cdot q\cdot q^6t^4$	$(K(3),J(2))$ $q^2\cdot q\cdot q^7t^5$	$(K(4),J(2))$ $q^2\cdot q\cdot q^9t^6$	$(K(5),J(2))$ $q^2\cdot q\cdot q^{11}t^7$	$\rightarrow \cdots$
$(J(3),K(2))$ $q\cdot q\cdot q^7t^5$	$(J(3),K(3))$ $q^2\cdot q\cdot q^8t^6$	$(J(3),K(4))$ $q^2\cdot q\cdot q^{10}t^7$	$(J(3),K(5))$ $q^2\cdot q\cdot q^{12}t^8$	$\rightarrow \cdots$
$(K(3),J(3))$ $q^2\cdot q\cdot q^8t^6$	$(K(4),J(3))$ $q^2\cdot q\cdot q^9t^7$	$(K(5),J(3))$ $q^2\cdot q\cdot q^{11}t^8$	$(K(6),J(3))$ $q^2\cdot q\cdot q^{13}t^9$	$\rightarrow \cdots$
$\downarrow$ $\vdots$	$\downarrow$ $\vdots$	$\downarrow$ $\vdots$	$\downarrow$ $\vdots$	

Our generating function is a sum of rational functions. Each orbit (marked by its box) contributes one. There are 2^d diagrams like this. Adding them all up proves our rationality theorem [HJ23]!

Results and conjectures for cusp $\{x^2 = y^3\}$

- Our rationality theorem [HJ23] does not require knowing the motive of each stratum.
- But when $d \leq 3$, we do know, and our method computes $Q_{d,X}(t)$ explicitly when $d \leq 3$. For example,

$$Q_{3,X}(t) = \frac{1 + (\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5)t^2 + (\mathbb{L}^6 + \mathbb{L}^7 + \mathbb{L}^8)t^4 + \mathbb{L}^9 t^6}{(1 - \mathbb{L}t)(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^3 t)}.$$

- We observe a **functional equation** for $d \leq 3$ by inspecting the symmetry. We conjecture it holds for all d.
- Observed patterns in $d \geq 3$, if true for all d, uniquely determine a conjectured formula for all $Q_{d,X}(t)$, which would then imply a strikingly simple formula for $\widehat{Z}_X(t)$ via a theorem about the forgetful map $\mathrm{Quot}_{d,n}(X) \to \mathrm{Coh}_n(X)$. The formula would confirm the **"rationality"** for $\widehat{Z}_X(t)$ and give the matrix count $\#\{AB = BA, A^2 = B^3\}$ in $\mathrm{Mat}_n(\mathbb{F}_q)$!

Current and future work

- (Work with Jiang in progress; using lattice zeta functions) Conjecture: for the $x^2 = y^3$ cusp, the numerator of $Q_{d,X}(t)$ is

$$\sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_{\mathbb{L}} (\mathbb{L}^d t^2)^j, \quad \begin{bmatrix} d \\ j \end{bmatrix}_{\mathbb{L}} = [\mathrm{Gr}(j,d)] \text{ the } q\text{-binomial coef.}$$

  The numerator of $\widehat{Z}_X(t)$ is $\sum_{n=0}^{\infty} \frac{\mathbb{L}^{-n^2} t^{2n}}{(1-\mathbb{L}^{-1})\dots(1-\mathbb{L}^{-n})}$.
- (Work with Jiang in progress; using lattice zeta functions) Conjecture: If X has only unibranched singularities, then $Q_{d,X}(t)$ is rational in t.
- (Work with Jiang in preparation; using Gröbner bases) For the node $X = \{xy = 0\}$, we have a recursive formula for $Q_{d,X}(t)$.
- (Hard) Conjecture: If X is a Gorenstein and projective curve with arithmetic genus g_a, then we have the functional equation

$$Q_{d,X}(t) = (\mathbb{L}^{d^2} t^{2d})^{g_a - 1} Q_{d,X}(\mathbb{L}^{-d} t^{-1}).$$