q-series from counting matrix points

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Introduction

An *(affine) variety* over $\mathbb{Z}$ is a system of polynomial equations $f_1(T_1,\dots,T_m)=\cdots=f_r(T_1,\dots,T_m)=0$ with integer coefficients. Counting *finite-field points* on varieties is of fundamental importance in number theory and arithmetic geometry.

Question. Can we count matrix points?

Definition (Matrix point)**.** Given $n\in\mathbb{Z}_{\geq 1}$ and finite field $\mathbb{F}_q$. A $\mathrm{Mat}_n(\mathbb{F}_q)$*-point* on the said variety is a tuple of pairwise commuting matrices $\underline{A}=(A_1,\dots,A_m)$ in $\mathrm{Mat}_n(\mathbb{F}_q)$ such that $f_j(\underline{A})=O_{n\times n}$ for all j.

Short answer. Yes, for smooth curves, smooth surfaces, and some singular curves(!!!). And partitions appear in the formulas.

Example: matrix points on plane

Theorem (Feit–Fine, '60).

$$\sum_{n\geq 0}\frac{|\{(A,B)\in\mathrm{Mat}_n(\mathbb{F}_q)^2 : AB=BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|}t^n=\prod_{i,j\geq 1}\frac{1}{1-t^iq^{2-j}}$$

Smooth varieties and Sato–Tate

Using geometric argument, one can bootstrap from Feit–Fine and get

Theorem 1 (H.)**.** *If the variety is smooth of dim ≤ 2 over $\mathbb{F}_q$, then its matrix point counts are determined by (usual) point counts over finite extensions of $\mathbb{F}_q$.*
More precisely, the analogous generating function counting its matrix points is an explicit infinite product in its zeta function.

This allows to lift deep theorems about finite-field point counts to matrix point counts. For example, consider the Legendre elliptic curve

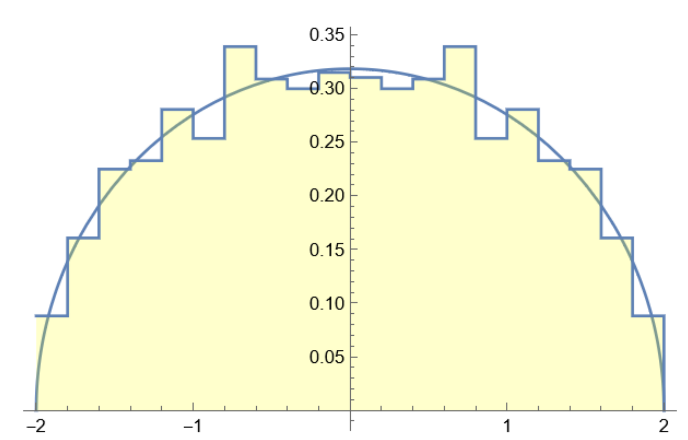
$$E_\lambda : y^2=x(x-1)(x-\lambda),\ \lambda\neq 0,1$$

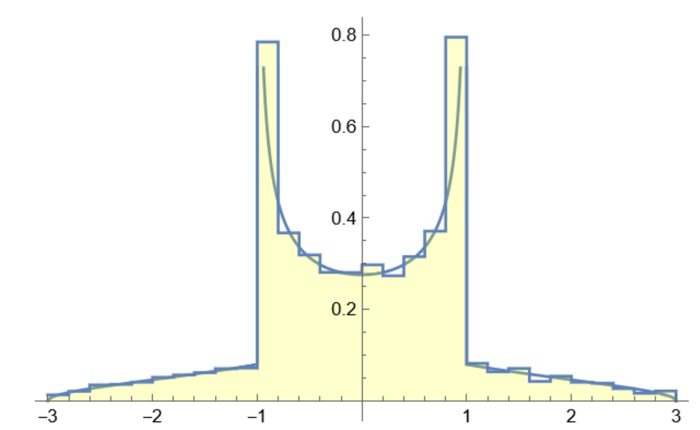
and the Alghren–Ono–Penniston $K3$ surface

$$X_\lambda : s^2=xy(x+1)(y+1)(x+\lambda y),\ \lambda\neq 0,-1.$$

They are special, for their finite-field point counts are given by *finite-field hypergeometric functions* ${}_2F_1^{\mathrm{ff}}$ and ${}_3F_2^{\mathrm{ff}}$ respectively, and as $\lambda\in\mathbb{F}_q$ varies, the *normalized error terms* in the point counts follow Sato–Tate-type distributions (Ono–Saad–Saikia).

Theorem 2 (H.–Ono–Saad)**.** *For any fixed $n\in\mathbb{Z}_{\geq 1}$, analogous statements hold for $\mathrm{Mat}_n(\mathbb{F}_q)$-points on E_λ and X_λ. In addition, the explicit formulas involve partitions of size up to n.*





Theorem (Blaser–Bradley–Vargas–Xing)**.** $\mathrm{Mat}_n(\mathbb{F}_p)$*-point counts follow analogous distributions for fixed E_λ, X_λ as we vary p.*

Singular curves and Rogers–Ramanujan

Consider the singular curve $C_k : Y^2=X^k$ for $k\in\mathbb{Z}_{\geq 2}$.

Theorem 3 (H.–Jiang)**.** *A series $f_k(q,t)\in\mathbb{Z}[[q^{-1},t]]$ encoding matrix point counts on C_k satisfies (i) $f_k(q,t)$ converges for $|q|>1, t\in\mathbb{C}$; (ii) $f_k(q,1)$ is a modular function in τ with $q^{-1}=e^{2\pi i\tau}$.*

The content is two-fold. On one hand, by giving $f_k(q,t)$, we explicit answer a wild linear algebra question: count solutions to $AB=BA, B^2=A^k$ for $A,B\in\mathrm{Mat}_n(\mathbb{F}_q)$. On the other hand, the assertion that $f_k(q,1)$ equals a modular function amounts to a Rogers–Ramanujan-type identity.

Example 4 (Andrews–Gordon from odd k)**.** For $k=2m+1$,

$$f_k(q,t)=\sum_{\lambda:\lambda_1\leq m}\frac{t^{2|\lambda|}}{q^{\sum_{i\geq 1}\lambda_i'^2}\prod_{i\geq 1}(q^{-1};q^{-1})_{\lambda_i'-\lambda_{i+1}'}},$$

so by an Andrews–Gordon identity

$$f_k(q,1)=\prod_{n\not\equiv 0,\pm(m+1)\pmod{2m+3}}(1-q^{-n})^{-1}.$$

Example 5 (New identity from even k)**.** Let $g^\lambda_\mu(p)$ be the Hall polynomial counting type-μ subgroups of an abelian p-group of type λ.

$$f_{2m}\left(\frac{1}{q},t\right)=(qt;q)_\infty^2\sum_{\substack{\lambda,\mu\\ \lambda_1\leq m}}\frac{q^{\sum_{i\geq 1}\lambda_i'^2}g^\lambda_\mu(q^{-1})\,(q;q)_{\lambda_m'}t^{2|\lambda|-|\mu|}}{(q;q)_{\mu_m'}(qt;q)^2_{\lambda_m'}\prod_{i\geq 1}(q;q)_{\lambda_i'-\lambda_{i+1}'}}$$

and we prove the identity (*indirectly; no direct proof yet!*)

$$f_{2m}(q,1)=1. \tag{1}$$

Conjecture. Analogue of Theorem 3 holds for all planar curves.

If true, the conjecture would produce a new framework for Rogers–Ramanujan identities: take a planar singularity, find the series $f(q,t)$ by counting matrix points, and get

sum side $:=f(q,1)=$ product side $:=$ modular function.