RESEARCH STATEMENT: YIFENG HUANG

Being primarily an arithmetic geometer, I research in the areas of mathematics inspired by Weil, Ramanujan, Cohen, Lenstra, and Andrews, and my works have involved Diophantine equations, partitions, q-series, hypergeometric functions, elliptic curves, modular forms, moduli spaces, random matrices, and planar singularities. I have initiated several new research directions, some of which start from a natural idea of "Diophantine equations over noncommutative rings" and draw inspirations from Cohen–Lenstra heuristics, Rogers–Ramanujan identities, Mordell–Lang conjecture, Sato–Tate conjecture, and a recent and surprising Oblomkov–Rasmussen–Shende conjecture that connects algebraic geometry and knot theory. One of my recent progresses hints a new arithmetic geometric framework of the Rogers–Ramanujan identities that appeals to mathematicians in partition theory, algebraic combinatorics, representation theory, mathematical physics, and knot theory.

1. Counting matrix points on varieties

Solutions to Diophantine equations, or rational points on varieties, are at the heart of mathematics, as evidenced by a long list of important theorems and related topics: Fermat's Last Theorem, Faltings' Theorem; modular forms, Galois representations, etc. A natural extension of such to the matrix setting, however, has been mostly unexplored, not because of its lack of importance, but probably due to its difficulty. More precisely, given an affine variety X defined by the Diophantine equation $f_1(T_1,\dots,T_m)=\dots=f_r(T_1,\dots,T_m)=0$ over any field k and given an integer $n\geq 1$, we consider the set $X(\mathrm{Mat}_n(k))$ of $n\times n$**-matrix points** on X defined by the matrix equation

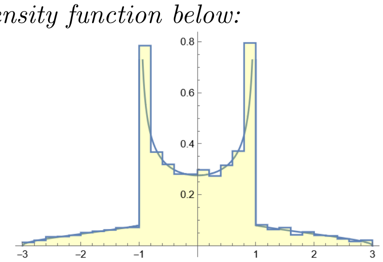
$$A_1,\dots,A_m\in \mathrm{Mat}_n(k): A_iA_j=A_jA_i \text{ and } f_j(A_1,\dots,A_m)=0 \text{ for } 1\leq j\leq r,$$

where the substitution $f_j(A_1,\dots,A_m)$ is well-defined precisely because the matrices are required to commute. These matrix equations have natural connections to algebraic geometry and mathematical physics [4, 8], and arithmetic statistics à la Cohen and Lenstra [15]. However, few examples of such equations are understood for general n.

I have developed an approach to study such matrix equations in general. More precisely, when k is a finite field, using ideas inspired by Cohen and Lenstra, I established an Euler product formula (**Theorem 0**) on matrix point counts [25], one of whose implications is that *to study matrix point counts in general it suffices to study much fewer examples.* Using this and Feit–Fine theorem, I obtained an explicit formula for matrix point counts on any smooth curve or smooth surface in terms of its Weil zeta function. One application among potentially many more is as follows.

Theorem 1 (H.–Ono–Saad [28]). *For $\lambda\in\mathbb{F}_p\setminus\{0,1\}$, let $X_\lambda: s^2=xy(x+1)(y+1)(x+\lambda y)$ be the Ahlgren–Ono–Penniston $K3$ surface [1]. Then for fixed $n\geq 1$, a suitably normalized distribution of $\#X_\lambda(\mathrm{Mat}_n(\mathbb{F}_p))$ as λ varies approaches a universal distribution as $p\to\infty$, with probability density function below:*

$$f(t):=\begin{cases}\sqrt{\frac{3-|t|}{1+|t|}} & \textit{if } 1<|t|\leq 3,\\ \sqrt{\frac{3-t}{1+t}}+\sqrt{\frac{3+t}{1-t}} & \textit{if } |t|<1,\\ 0 & \textit{otherwise.}\end{cases}$$



In the same vein, we proved an analogous result for the Legendre family of elliptic curves (**Theorem 1'**), where the limiting distribution is the semicircular distribution [28], directly analogous to the Sato–Tate conjecture established by Taylor and his collaborators [3, 14, 22, 37], a major breakthrough in this century. Both of our results above fit into the "vertical" regime of the Sato–Tate conjecture, whence in Sato–Tate we fix our variety over $\mathbb{Q}$ and vary the prime ("horizontal"). The potential of Theorem 0 is further evidenced by an REU paper [7] on the horizontal side of Theorems 1 and 1'. In an ongoing project with Ono, I explore the matrix point counts of 3-folds under the horizontal regime. This would require a new understanding on commuting triples of matrices. So far, I have proved that for certain rigid Calabi–Yau 3-folds Y and $n\leq 3$, the counts $\#Y(\mathrm{Mat}_n(\mathbb{F}_p))$ are given by modular forms and satisfy Sato–Tate with the semicircular distribution. A special case takes an elegant and concrete form as follows.

Theorem 2. *For an odd prime p, let a_p be the p-th coefficient of the modular form $q\prod_{n\geq 1}(1-q^{2n})^4(1-q^{4n})^4$. Then for each $n\leq 3$, there is an explicit polynomial $f_n(q,a)\in\mathbb{Z}[q,a]$ of degree n in a such that for any odd prime p, the value $f_n(p,a_p)$ gives the number of solutions to the matrix equation*

$$A_1,\dots,A_4\in \mathrm{GL}_n(\mathbb{F}_p): A_iA_j=A_jA_i \textit{ and } A_1+A_1^{-1}+\dots+A_4+A_4^{-1}=0.$$

2. Rogers–Ramanujan via matrix points on singular curves

The famous Rogers–Ramanujan (RR) identities are two identities in q-series and integer partitions that have been since connected to a wide range of mathematics, such as K-theory, vertex operator algebras, knot theory, and conformal field theory. An influential framework by Griffin, Ono, and Warnaar [21], from the perspective of affine Kac–Moody algebras and Hall–Littlewood polynomials, generates infinite families of generalizations of RR identities featuring summations over partitions with parts not exceeding m. My research in matrix points on singular curves is leading to a new arithmetic geometric frontier in the theory of RR identities, which also connects to Cohen–Lenstra's work and knot theory. Some simplest examples conjecturally give a new infinite family of RR-type identities.

To illustrate the flavor of RR-type identities in my work in a way that foreshadows its connection to Cohen–Lenstra's work, consider an interpretation of the Andrews–Gordon (AG) identities as follows. For an integer partition λ, let $|\lambda|$ denote its size and λ_i (*resp.* λ_i') denote the size of its i-th row (*resp.* column) in the Young diagram of λ. The polynomial $a_q(\lambda) := q^{\sum \lambda_i'^2} \prod_{i\geq 1} (q^{-1};q^{-1})_{\lambda_i' - \lambda_{i+1}'}$ famously counts the number of automorphisms of an abelian p-group of type λ when $q = p$ is a prime [31]. As usual, let $(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1})$, $(a;q)_\infty := (1-a)(1-aq)\dots$, and $(a_1,\dots,a_k;q)_n := (a_1;q)_n \dots (a_k;q)_n$, as is for $n=\infty$. Then two special cases of the AG identities can be compactly restated as the following sum over partitions with parts not exceeding m:

$$\sum_{\lambda_1 \leq m} \frac{t^{|\lambda|}}{a_{q^{-1}}(\lambda)} \Bigg|_{t=q^\sigma} = \frac{(q^{m+1-\sigma m}, q^{m+2+\sigma m}, q^{2m+3}; q^{2m+3})_\infty}{(q;q)_\infty}, \quad \sigma = 0, 1, \tag{1}$$

which specializes to the RR identities when $m = 1$. The presence of $1/\#\operatorname{Aut}$ here is a pivotal feature of the Cohen–Lenstra heuristics in arithmetic statistics [15], random partitions [18], and random p-adic matrices [17, 38, 39].

My framework introduced in [25] can be described as follows. Let $X = \operatorname{Spec} A$ be an affine variety over $\mathbb{F}_q$. Define

$$\widehat{Z}_X(t) := 1 + \sum_{n=1}^\infty \frac{\# X(\operatorname{Mat}_n(\mathbb{F}_q))}{\# \operatorname{GL}_n(\mathbb{F}_q)} t^n = \sum_M \frac{t^{\dim_{\mathbb{F}_q} M}}{\# \operatorname{Aut}(M)} \in \mathbb{Q}[[t]],$$

where M ranges over A-modules with $\dim_{\mathbb{F}_q} M < \infty$ up to isomorphism. The equivalence of the definitions is established in [25] and the second definition reveals that it directly generalizes Cohen–Lenstra's zeta function in [15]. If X is a reduced curve with exactly one singularity (whose germ is given by the complete local ring R), then the following series involving the normalization $\widetilde{X}$ of X depends only on R, and shall be the focus object of study:

$$N\widehat{Z}_R(t) := \widehat{Z}_X(t) / \widehat{Z}_{\widetilde{X}}(t).$$

The richness of $N\widehat{Z}_R(t)$ is already present in the simplest family of curve singularities: $R^{(2,n)} := \mathbb{F}_q[[x,y]]/(y^2 - x^n)$ (when $\mathbb{F}_q$ has characteristic two and $n = 2m$ is even, we shall replace the equation by $y(y-x^m) = 0$). Then $N\widehat{Z}_{R^{(2,2m+1)}}(t)$ essentially gives the left-hand side of (1), and $N\widehat{Z}_{R^{(2,2m)}}(t)$ gives a new series, see below.

Theorem 3 (H.–Jiang [26])**.** *For $m \geq 1$, we have*

$$N\widehat{Z}_{R^{(2,2m+1)}}(t) = \sum_{\lambda_1 \leq m} \frac{1}{a_q(\lambda)} t^{2|\lambda|} \in \mathbb{Z}[[q^{-1}, t]],$$

and

$$N\widehat{Z}_{R^{(2,2m)}}(t) = \sum_{\lambda_1 \leq m} \sum_{\mu \subseteq \lambda} \frac{g^\lambda_\mu(q)\,(q^{-1};q^{-1})_{\lambda_m'}}{a_q(\lambda)\,(q^{-1};q^{-1})_{\mu_m'}} t^{2|\lambda| - |\mu|} (q^{-\lambda_m' - 1} t; q^{-1})_\infty^2 \in \mathbb{Z}[[q^{-1}, t]],$$

where $g^\lambda_\mu(q) := \sum_\nu g^\lambda_{\mu\nu}(q)$ is a sum of the Hall polynomials [31] that can also be interpreted as the principal specialization of a skew Hall–Littlewood function.

While the two-variable series $N\widehat{Z}_{R^{(2,2m)}}(t)$ given as a sum of partitions with bounded parts is not known to fit into an existing framework, its relevance to the theory of RR identities is evidenced by the following theorem-conjecture. Note that (1) with $\sigma = 0$ implies that $N\widehat{Z}_{R^{(2,2m+1)}}(\pm 1)$ is an infinite product. The following says the same for $R^{(2,2m)}$.

Theorem-Conjecture 4 (H.–Jiang [26])**.** *Let $m \geq 1$. Then we have $N\widehat{Z}_{R^{(2,2m)}}(1) = 1$ and we conjecture that*

$$N\widehat{Z}_{R^{(2,2m)}}(-1)|_{q \mapsto q^{-1}} \overset{?}{=} \frac{(q^2;q^2)_\infty (q^{m+1};q^{m+1})_\infty^2}{(q;q)_\infty^2 (q^{2m+2};q^{2m+2})_\infty}.$$

The case $m = 1$ is known to my previous work [25], while the case $m \geq 2$ is based on numerical evidences. We note that like (1), the conjectural right-hand side of this new family of RR identities also gives a modular function.

3. Quot schemes on singular curves and high-rank Oblomkov–Rasmussen–Shende

The theorems in the previous section are consequences of my new results in enumerative algebraic geometry on a topic with rich connections to algebraic combinatorics, geometric representation theory, mathematical physics, and knot theory. The last decade has seen exciting developments on the Hilbert schemes of points on singular plane curves and their surprising connections to link homology and q,t-Catalan numbers [19, 20, 32, 34], exemplified by the Oblomkov–Rasmussen–Shende (ORS) conjecture [33] and the Hilb-vs-Quot conjecture [29]. Joint with Jiang [26, 27], we advanced a high-rank generalization of this subject.

Let k be a field and let R be the complete local ring of a k-variety at a k-point. For a finitely generated R-module E and $n \geq 0$, consider the Quot scheme $\mathrm{Quot}_{E,n}$ parametrizing R-submodules of E with k-codimension n. When $E = R$, $\mathrm{Quot}_{E,n}$ is the Hilbert scheme of n points on the germ $\operatorname{Spec} R$. The high-rank case $E = R^d := R^{\oplus d}$ for $d \geq 1$ thus gives an infinite family of generalizations of the Hilbert scheme of points. As a geometric analogue of point counts over finite fields, we consider a motivic generating function in the Grothendieck ring of k-varieties:

$$\mathcal{Z}_E^R(t) = \mathcal{Z}_E(t) := \sum_{n\geq 0} [\mathrm{Quot}_{E,n}]\, t^n \in K_0(\mathrm{Var}_k)[[t]],$$

where the class $[\mathrm{Quot}_{E,n}]$ can be understood as a measure of this moduli space up to cut-and-paste.

When R is the germ of a reduced curve and E is a torsion-free R-module of rank d, we proved two general results. Let $\widetilde{R}$ be the normalization of R and assume $\widetilde{R} = k[[T]]^{\times s}$, where s can be understood as the branching number.

Theorem 5 (H.–Jiang [26])**.** *In the setting above,*

(a) (Rationality) *The series* $\mathcal{NZ}_E(t) := \mathcal{Z}_E(t)/\mathcal{Z}^R_{\widetilde{R}^d}(t) = (t;\mathbb{L})_d^s \mathcal{Z}_E(t)$ *is a polynomial in* t*, where* $\mathbb{L} = [\mathbb{A}^1]$.

(b) (Functional Equation) *If* R *is a plane curve germ with Serre invariant* $\delta := \dim_k \widetilde{R}/R$*, then we have*

$$\mathcal{NZ}_{R^d}(t) \approx (\mathbb{L}^{d^2} t^{2d})^\delta \mathcal{NZ}_{R^d}(\mathbb{L}^{-d}t^{-1}) \tag{2}$$

up to point counts over finite fields.

Part (a) is a common generalization of Bifet [6] ($d \geq 1$, R smooth) and Bejleri–Ranganathan–Vakil [5] ($d = 1$, R singular). In fact, a stronger *relative* result holds (**Theorem 5"(a)**). For part (b), we conjecture that (2) holds in $K_0(\mathrm{Var}_k)[t]$ (**Conjecture 5'**), which generalizes a well-known functional equation in the $d = 1$ case that reflects the Serre duality [35]. Attacking Conjecture 5' for $d \geq 2$ would likely require a deep geometric understanding of the high-rank analogue of the compactified Jacobian, which is no longer a scheme, but a stack. The point-count statement in part (b) is derived from a more general theorem that holds for any curve germ, where R^d needs to be replaced by the d-fold sum of the dualizing module (**Theorem 5"(b)**). It is a $d \geq 1$ generalization of Yun [40], and can be viewed as a new result in the theory of lattice zeta functions pioneered by Solomon, Bushnell, and Reiner [9, 36].

We also computed $\mathcal{NZ}_{R^d}(t)$ and $\mathcal{NZ}^R_{\widetilde{R}^d}(t)$ for $R = R^{(2,n)} := k[[x,y]]/(y^2 - x^n)$, where $n \geq 2$. Recall the notation $g^\lambda_\mu(q)$ from Theorem 3, and let (m^d) denote the partition consisting of d copies of m.

Theorem 6 (H.–Jiang [26])**.** *Let* $m \geq 1$*,* $d \geq 0$*, then we have the following in* $\mathbb{Z}[\mathbb{L}, t]$.

(a) *For* $R = R^{(2,2m+1)}$*, we have* $\mathcal{NZ}^R_{\widetilde{R}^d}(t) = \sum_{\mu\subseteq(m^d)} g^{(m^d)}_\mu(\mathbb{L})\, (\mathbb{L}^d t)^{|\mu|}$*, and* $\mathcal{NZ}_{R^d}(t) = \mathcal{NZ}^R_{\widetilde{R}^d}(t^2)$.

(b) *For* $R = R^{(2,2m)}$*, we have* $\mathcal{NZ}^R_{\widetilde{R}^d}(t) = \sum_{\mu\subseteq(m^d)} g^{(m^d)}_\mu(\mathbb{L})\, (\mathbb{L}^d t)^{|\mu|} \frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_d}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{d-\mu_1'}}$*, and*

$$\mathcal{NZ}_{R^d}(t) = \sum_{\lambda\subseteq(m^d)} \sum_{\mu\subseteq\lambda} g^{(m^d)}_\lambda(\mathbb{L})\, g^\lambda_\mu(\mathbb{L})\, (t;\mathbb{L})^2_{d-\lambda'_m} t^{|\lambda|} (\mathbb{L}^d t)^{|\lambda|-|\mu|} \frac{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\lambda'_m}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_{\mu'_m}}. \tag{3}$$

These computations suggest a new framework of bounded analogues of the (skew) Cauchy identity for Hall–Littlewood polynomials, featuring sums over partitions bounded by a rectangle. To elaborate, the computation of $\mathcal{NZ}_{R^d}(t)$ in Part (a) requires a new identity (**Lemma 6'**) for a fixed μ, m, d:

$$\sum_{\lambda:\ \mu\subseteq\lambda\subseteq(m^d)} g^{(m^d)}_\lambda(q)\, g^\lambda_\mu(q)\, t^{|\lambda|} (t;q)_{d-\lambda'_m} = g^{(m^d)}_\mu(q)\, t^{|\mu|},$$

and Part (b) and the functional equation (Theorem 5(b)) imply a new identity (**Corollary 6"**) for each m, d:

The polynomial $f(t,q)$ *defined by* (3) *with* $\mathbb{L} \mapsto q$ *satisfies* $f(t,q) = (q^{d^2} t^{2d})^m f(q^{-d}t^{-1}, q)$.

When $m \to \infty$, both identities are direct consequences of the skew Cauchy identity for $P_\mu(x_1, \dots, x_n; z)$. Lemma 6' is proved by q-hypergeometric transformations, while Corollary 6" is not known to have a direct proof.

Theorem 6 can also be viewed as an initial exploration of a more general framework in high ranks. When R is a planar singularity and $d = 1$, the ORS conjecture [33] and the Hilb-vs-Quot conjecture [29] postulate precise connections between $\mathcal{NZ}_{R^d}(t)$, $\mathcal{NZ}_{\widetilde{R}^d}(t)$, and the Khovanov–Rozansky homology of the algebraic link associated with R. Our findings suggest that when $d > 1$, the connections must take a more refined form. More precisely, the Hilb-vs-Quot conjecture can be reformulated as $\mathcal{NZ}_{R^d}(t) = \mathcal{NZ}_{\widetilde{R}^d}(t)|_{\mathbb{L}\mapsto\mathbb{L}t}$ if $d = 1$, but Theorem 6(a) says $\mathcal{NZ}_{R^d}(t) = \mathcal{NZ}_{\widetilde{R}^d}(t)|_{t\mapsto t^2}$ if $R = R^{(2,2m+1)}$ and $d \geq 1$. This means neither of the rules generalizes; in fact, they do not contradict when $R = R^{(2,2m+1)}$ and $d = 1$ only because $\mathcal{NZ}_{\widetilde{R}}(t) = 1 + \mathbb{L}t + \cdots + (\mathbb{L}t)^m$.

The huge potential of considering a high-rank ORS is evidenced by a surprising connection between $\mathcal{NZ}_{R^d}(t)$ and $\widehat{Z}_R(t)$ in §2, which we call the "rank $\to \infty$ limit" theorem. Let R be *any* k-variety germ, and $\widehat{\mathcal{Z}}_R(t) \in K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}, t]]$ be the motivic generating function for the stack of R-modules of finite k-dimension. It is the motivic refinement of $\widehat{Z}_R(t)$ in §2. Equip $K_0(\mathrm{Var}_k)[[\mathbb{L}^{-1}]]$ with the topology induced from the dimension filtration.

Theorem 7 (H.–Jiang [26]). *We have* $\lim_{d\to\infty} \mathcal{Z}_{R^d}(\mathbb{L}^{-d}t) = \widehat{\mathcal{Z}}_R(t)$ *coefficientwise.*

4. Further topics

4.1. Noncommutative unit equations, Mordell–Lang, and arithmetic dynamics. One version of the celebrated S-unit theorem [30] states that $x+y = 1$ cannot have infinitely many solutions in a finitely generated subgroup of $\mathbb{C}^\times$. In [24], I gave the first noncommutative analogue by proving an effective finiteness theorem for the unit equation on certain subsemigroups of $\mathbb{H}^\times$, the multiplicative group of the quaterions (**Theorem 8**). It naturally applies to arithmetic dynamics since endomorphism rings of elliptic curves can be embedded into quaternions. In this direction, I proved an "infinite intersection implies common iterate" theorem on genus-one curves over any field (**Theorem 9**). Unit equations fit into the context of Mordell–Lang conjecture, i.e., finiteness assertions about the intersection of a finitely generated subgroup and a subvariety of an algebraic group. To prove Theorem 8, I proved a noncommutative analogue of Mordell–Lang in the real algebraic group $\mathbb{H}^\times$ (**Theorem 10**).

4.2. Configuration spaces of points on Riemann surfaces. Let $\mathrm{Conf}_n(X)$ denote the configuration space of n distinct unlabeled points on X. A beautiful observation in the theory of representation stability by Church, Ellenberg, and Farb [13] links two celebrated arrays, the Betti numbers of $\mathrm{Conf}_n(\mathbb{C})$ (due to Arnol'd [2]) and the count of $\mathrm{Conf}_n(\mathbb{F}_q)$ (monic square-free polynomials over $\mathbb{F}_q$):

$$\begin{array}{lll} n = 0,1, \quad h^0(\mathrm{Conf}_n(\mathbb{C})) = 1 & \longleftrightarrow & \#\,\mathrm{Conf}_n(\mathbb{F}_q) = q^n, \\ n \geq 2, \quad h^0(\mathrm{Conf}_n(\mathbb{C})) = 1, h^1(\mathrm{Conf}_n(\mathbb{C})) = 1 & \longleftrightarrow & \#\,\mathrm{Conf}_n(\mathbb{F}_q) = q^n - q^{n-1}, \end{array}$$

using a "purity of weight $2i$" statement in mixed Hodge theory. Joint with Cheong, I discovered a surprising genus 1 analogue with purity of weight $\lfloor 3i/2 \rfloor$. For $g, r \geq 0$, let $\Sigma_{g,r}$ be an r-punctured Riemann surface of genus g.

Theorem 11 (Cheong–H. [11]). $\sum_{n,i\geq 0} (-1)^i h^i(\mathrm{Conf}_n(\Sigma_{1,1}))u^{2n-w(i)}t^n = \dfrac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2}$, *where* $w(i) = \lfloor \dfrac{3i}{2} \rfloor$, *and the mixed Hodge structure of* $H^i(\mathrm{Conf}_n(\Sigma_{1,1}))$ *is pure of weight* $w(i)$.

For more general $g, r \geq 1$, the purity no longer holds. Nevertheless, I proved in [23] an S_n-equivariant and mixed Hodge theoretic result that compares the *ordered* configuration spaces of X and $X - \{\mathrm{pt}\}$, where X is a complex variety of arbitrary dimension (**Theorem 12**). This implies a rational generating function for the mixed Hodge numbers when $g = 1$.

Corollary 13 (combine [11, 23]). *For* $r \geq 1$, *we have*

$$\sum_{p,q,i,n\geq 0} h^{p,q;i}(\mathrm{Conf}_n(\Sigma_{1,r}))x^p y^q (-u)^i t^n = \frac{1}{(1+xyut)^{r-1}} \frac{1-(x+y)ut + (x+y)xyu^2t^3 - x^2y^2u^3t^4}{(1-t)(1-x^2yu^2t^2)(1-xy^2u^2t^2)}.$$

4.3. Cohen–Lenstra via random p-adic matrices. In random matrix theory, universality refers to distributional behaviors as the size of the random matrix approaches ∞. The Cohen–Lenstra distribution and its analogues arise as the universal distribution of the cokernel of p-adic matrices in several random models, some of which are relevant to arithmetic statistics, random partitions, and graph theory [16, 38, 39]. Joint with Cheong [10, 12], I studied two new models that expand the inventory of Cohen–Lenstra distributions: (i) polynomial $P(X)$ of a random p-adic matrix X; (ii) random p-adic matrix with a fixed residue modulo p. These settings are novel: in (i) the entries are intricately dependent unlike most known models, and in (ii) an important condition of "ϵ-balancedness" is violated. The model (i) is of special interest because of a structural constraint it induces: $\mathrm{cok}\, P(X)$ is canonically a module over $R := \mathbb{Z}_p[t]/P(t)$. One may combine (i)(ii), namely, taking $P(X)$ for a random p-adic matrix X with a fixed residue. In [12], we showed that when $P(t)$ is monic, the Cohen–Lenstra distribution obtained this way is a new distribution on finite R-modules that depends on zeroth and first Betti numbers in the sense of minimal resolutions (**Theorem 14**). This distribution generalizes known random partition models in a fundamentally new direction, namely, from finite $\mathbb{Z}_p$-modules, which are classified by partitions, to finite R-modules, which are not classified by partitions.

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