Vector Fields and Cohomology

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The Bott Residue Theorem [B67] (see also [C03]) says that a holomorphic vector V with simple isolated zeroes on a compact complex manifold M determines the Chern numbers of M as sums of residues over the zero set Z of V determined by how V behaves near Z. Note, a holomorphic vector field V on M is a holomorphic section of the holomorphic tangent bundle of M. A zero x of V is simple if on a neighbourhood U of x, $\dim C[U]/(a_1, \dots, a_n) = 1$, where $a_1, \dots, a_n$ are the coefficients of V on U. The simplest condition which implies V has only simple isolated zeros Z is that V is generated by a G_m-action $\lambda : \mathbb{C}^* \to Aut(M)$ with isolated fixed point set Z.

The BRF has turned out to be important in enumerative geometry, for example in work of Ellingsrud-Stromme [JAMS96] and Konsevitch. Bott's proof inspired the following generalization, which follows from considering the spectral sequence associated to the contraction operator $i(V) : \Omega^p \to \Omega^{p-1}$. The crucial fact is that if M is a compact Kaehler manifold and Z is non trivial, then this spectral sequences degenerates at d_1. That is, $d_i = 0$ for all i. For defininiteness, we will assume our holomorphic vector field arises from a G_m-action with fixed points.

Theorem 1 (CL73,CL77)**.** *Let M be a compact Kaehler manifold of dimension n having a G_m-action with non trivial fixed point set Z. Then $H^p(M, \Omega^q) = 0$ if $|p - q| > \dim Z$. Suppose Z is finite. Then the Hodge decomposition theorem implies $H^*(M) = \bigoplus_{p\geq 0} H^{2p}(M) = \bigoplus_{p\geq 0} H^p(M, \Omega^p)$. Moreover, the set $\mathbb{C}^Z$ of $\mathbb{C}$-valued functions on Z has a filtration*

$$0 = F_{-1} \subset F_1 = \mathbb{C} \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^Z$$

such that $F_i F_j \subseteq F_{i+j}$ and there are graded algebra isomorphisms

$$H^*(M) = \bigoplus_{p\geq 0} H^{2p}(M) \cong \bigoplus_{p\geq 0} F_p/F_{p-1} = \mathrm{Gr}_F\, \mathbb{C}^Z.$$

If Z is finite, then M is necessarily projective since $H^2(M) = H^1(M, \Omega^1)$. So from now on, we will denote M by X.

Example 2. Let $X = \mathbb{P}^n$ and consider the G_m-action

$$t \cdot [z_0, \dots, z_n] = [t^{a_0} z_0, \dots, t^{a_n} z_n],$$

where all a_i are distinct. Then $Z = \{[1, 0, \dots, 0], \dots, [0, \dots, 0, 1]\}$. The element $f \in \mathbb{C}^Z$ defined by $f([e_i]) = a_i$ has the property that in the isomorphism above, f represents $c_1(H)$ in the associated graded, where H is a hyperplane section in $\mathbb{P}^n$. Moreover, $F_i = \mathbb{C} \oplus \mathbb{C}f \oplus \cdots \oplus \mathbb{C}f^{i-1}$. Note how the VanDerMonde determiinant plays a role.

When Z is infinite, we still get an interesting result when $H^p(X, \Omega^q) = 0$ for $p \neq q$. In that case, we also have $H^p(Z, \Omega^q) = 0$ for $p \neq q$, and the cohomology of X is computed on Z as follows:

Theorem 3 (ACL86)**.** *Suppose $H^p(X,\Omega^q) = 0$ if $p \neq q$. Then $H^p(Z,\Omega^r) = 0$ if $p \neq r$ and the graded algebra $H^*(Z)$ admits an increasing filtration $G.$ such that*

$$H^*(X) \cong \mathrm{Gr}_{G.}\, H^*(Z).$$

This theorem enables us to study G_m-stable subvarieties Y of X such that Y^{G_m} is finite

Theorem 4 (ACL86)**.** *Let Y be a Zariski closed G_m-stable subset of X such that Y^{G_m} is finite and the cohomology restriction maps $H^*(X) \to H^*(Y)$ and $H^0(X^{G_m}) \to H^0(Y^{G_m})$ are surjective. Note: the latter assumption implies no component of X^{G_m} contains more than one element of Y^{G_m}. Then the filtration of $H^*(X^{G_m})$ pulled back to $H^0(Y^{G_m})$ gives an isomorphism of graded rings*

$$\mathrm{Gr}\, H^0(Y^{G_m}) \to H^*(Y).$$

Schubert varieties and Springer varieties in a flag variety G/B are two interesting cases. Let $T \subset B \subset G$ be a maximal torus T in a Borel (that is, maximal connected solvable) subgroup B of a connected semisimple algebraic group G over $\mathbb{C}$. The G-homogeneous space G/B is a projective variety, and the Weyl group of T, defined as $N_G(T)/T$, is a finite reflection group such that $(G/B)^T = WB$ via the bijection $w \to wB$. Hence, $H^p(G/B, \Omega^q) = 0$ if $p \neq q$.

Example 5. The standard example has $G = GL(n,\mathbb{C})$, B the upper triangular elements of G and T the torus on the diagonal of G. Then G/B is the variety of all full flags in $\mathbb{C}^n$, and W is the set of all $n \times n$ permutation matrices. The T-fixed flags are the column flags of the permutation matrices.

The Schubert variety X_w ($w \in W$) is the Zariski closure of the B-orbit BwB. Schubert varieties are in general singular but are T-stable and $H^*(G/B) \to H^*(X_w)$ is always surjective. The T-fixed points in X_w define a Bruhat interval $[e,w]$ in W. Now consider a G_m-action $(S, G/B)$, where $S \subset T$ and $(G/B)^S = WB$, say $S = exp(s)$ where $s \in Lie(T)$ is regular. Thus, we get

Theorem 6 (ACL86)**.** *For any Schubert variety X_w in G/B and any $w \in W$,*

$$H^*(X_w) \cong \mathrm{Gr}\, \mathbb{C}^{[e,w]}.$$

Moreover, the filtration of $\mathbb{C}^{[e,w]}$ arises in a natural way. Since $\mathbb{C}^{[e,w]} \cong \mathbb{C}([e,w]\cdot s)$ and $[e,w]\cdot s$ is a closed subset of $Lie(T)$, we have a natural filtration $F_i \subset \mathbb{C}([e,w]\cdot s)$, namely, F_i is defined to be the elements which are restrictions of polynomials on $Lie(T)$ of degree at most i. Then this filtration is the image of the filtration of $\mathbb{C}^{[e,w]}$, and therefore we obtain a natural description

$$H^*(X_w) \cong \mathrm{Gr}\, \mathbb{C}([e,w]\cdot s).$$

In particular, G/B itself is a Schubert variety. Hence

$$H^*(G/B) \cong \mathrm{Gr}\, \mathbb{C}(W \cdot h)$$

for the natural filtration of $\mathbb{C}(W \cdot h)$, where $h \in Lie(T)$ is regular in the sense that the G_m defined by h has $G/B)^{G_m} = (G/B)^T$. From this one deduces without too much difficulty the classical description of $H^*(G/B)$ as the *coinvariant algebra* $\mathbb{C}(Lie)T)/I^W$, where I^W is the ideal generated by homogeneous W-invariants.

Here is a more intriguing application. A famous result of T. A. Springer says that if X is a Springer variety in G/B, then the Weyl group W of G acts on the cohomology algebra $H^*(X)$. The definition of Springer's action quite remarkable since it doesn't arise from an action of W on X. In addition, every irreducible representation of W is realized on the middle dimension of some Springer variety. We will explain below how in some important cases, the W-action on $H^*(X)$ can be described in a natural way using a certain G_m-action on G/B.

First note that by a fundamental result of Borel, G/B parameterizes the set $\mathbb{B}$ of all Borels in G via the bijection $gB \to gBg^{-1}$. Thus, we may identify the flag variety G/B with $\mathbb{B}$. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone consisting of the closed (and normal) variety of all nilpotents in $\mathfrak{g}$. G acts on $\mathcal{N}$ by conjugation with a unique orbit called the regular nilpotent orbit. If n is nilpotent, the Springer variety $\mathbb{B}_n \subset \mathbb{B}$ is defined to be the set of all Borels whose Lie algebra contains n. It is well known that $\mathbb{B}_n$ is closed in $\mathbb{B}$ and is connected for all n. Note that except for the obvious cases $n = 0$, where $\mathbb{B}_n = \mathbb{B}$, and n regular, where $\mathbb{B}_n$ is a point, $\mathbb{B}_n$ has more than one irreducible component.

Example 7. In the standard case $G = GL(n, \mathbb{C})$, a nilpotent whose J.C.F. has rank $n-2$ is called subregular. If n is subregular, then $\mathbb{B}_n \subset GL(n, \mathbb{C})/B$ is the union of n $\mathbb{P}^1$'s whose intersection pattern is determined by the (dual) Dynkin diagram of type A_n. Subregular nilpotents n are defined for arbitrary G, and if n is subregular, then $\mathbb{B}_n$ is a general Dynkin curve, roughly as described above.

We now apply the above ideas to give a description of Springer's action. To do so, let us aassume $G = GL(n, \mathbb{C})$. Then, by a result of N. Spaltenstein [Sp76], the inclusion $\mathbb{B}_n \subset \mathbb{B}$ induces a surjection $H^*(\mathbb{B}) \to H^*(\mathbb{B}_n)$ for all $n \in \mathcal{N}$. Moreover, for any $n \in \mathcal{N}$, there exists a semisimple $s \in \mathfrak{gl}(n, \mathbb{C})$ such that $[s, n] = 0$ and $n+s$ is regular in the sense that $n + s$ is contained in only finitely many Borel subalgebras. This gives a torus action $(S, \mathbb{B})$ stabilizing $\mathbb{B}_n$ such that $(\mathbb{B}_n)^S$ is finite, and morever $H^*(\mathbb{B}^S) \to H^*((\mathbb{B}_n)^S)$ is surjective. The interesting fact is that there exists a natural identification of $W \cdot s$ and $(\mathbb{B}_n)^S$. Hence one has $H^*((\mathbb{B}_n)^S) = \mathbb{C}(W \cdot s)$. Consequently, Theorem 4 gives

Theorem 8 (C86)**.** *Suppose $G = GL(n, \mathbb{C})$ and $n \in \mathcal{N}$. Then*

$$H^*(\mathbb{B}_n) \cong \mathrm{Gr}\,\mathbb{C}(W \cdot s),$$

where the filtration of $\mathbb{C}(W \cdot s)$ is the natural filtration by degree. Furthermore, the action of W on $H^*(\mathbb{B}_n)$ is induced by the natural action of W on $\mathbb{C}(Lie(T))$ given by $w \cdot f(x) = f(w^{-1} \cdot x)$.

This discussion brings up two **Problems**. First, what is the Poincaré polynomial of $\mathrm{Gr}\,\mathbb{C}(W \cdot s)$ for any Weyl group orbit $W \cdot s$ where $s \in Lie(T)$? Here W can be any Weyl group. A sub problem, which doesn't seem to be trivial, is to determine the ideal $I(W \cdot s) \subset \mathbb{C}(Lie(T))$ for arbitrary W and $s \in Lie(T)$. Secondly, describe all pairs (G, n) such that $H^*(\mathbb{B}) \to H^*(\mathbb{B}_n)$ is surjective.

The problem of finding the Poincaré polynomial of $H^*(\mathbb{B}_n)$ can also be attacked from by using a result of DeConcini, Lusztig and Procesi [DLP88], which says that $\mathbb{B}_n$ always has an affine paving. But this paving is hard to describe.

In our final movement, we will consider holomorphic vector fields with exactly one zero. Unipotent actions, that is G_a-actions, sometimes have this property. Note: the fixed point set of a unipotent group is always connected. A G_a-action on X is the algebraic action given by an algebraic homomorphism $\phi : \mathbb{C} \to Aut(X)$. The holomorphic vector field on X generated by a G_a is said to be algebraic. A G_a-action on X so that $\phi(\mathbb{C})$ has exactly one fixed point o gives a holomorphic vector field with unique zero o.

Here is an example.

Example 9. Suppose $X = \mathbb{P}^3$, and let $\phi : \mathbb{C} \to Aut(\mathbb{P}^3)$ be given by $\phi(s) = I_4 + sJ$, where J is the 4×4 Jordan block of maximal rank 3. Then ϕ is an algebraic homomorphism. Explicity,

$$\phi(s) \cdot [z_0, z_1, z_2, z_3] = [z_0 + sz_1, z_1 + sz_2, z_2 + sz_3, z_3].$$

Note $o = [1, 0, 0, 0]$ is ϕ's unique fixed point. Now the algebraic vector field V generated by ϕ has local expansion on the affine open $z_0 \neq 0$ given by

$$V = (u_2 - u_1^2)\frac{\partial}{\partial u_1} + (u_3 - u_1u_2)\frac{\partial}{\partial u_2} - u_1u_3\frac{\partial}{\partial u_3},$$

using the standard affine coordinates $u_i = z_i/z_0$ for $i = 1, 2, 3$. Clearly, its unique zero o is not a simple zero, but we can consider o as the punctual scheme in X defined by the ideal $I(V) = (u_2 - u_1^2, u_3 - u_1u_2, u_1u_3) \subset \mathbb{C}[u_1, u_2, u_3]$. Notice that the coordinate ring $\mathcal{A} = \mathbb{C}[u_1, u_2, u_3]/I(V)$ of the punctual scheme o is isomorphic with $\mathbb{C}[u_1]/((u_1)^4)$, which is in fact isomorphic with $H^*(\mathbb{P}^3)$.

It remains to explain why $\mathcal{A}$ has a grading. To do so, we introduce the notion of a (G_a, G_m) pair on X. Such a pair consists of algebraic one parameter groups $\phi : \mathbb{C} \to Aut(X)$ and $\lambda : \mathbb{C}^* \to Aut(X)$ such that $\lambda(t)\phi(s)\lambda(t)^{-1} = \phi(t^2 s)$ for all s, t. A (G_a, G_m) pair is clearly equivalent to an algebraic action of the upper triangular (Borel) subgroup $\mathfrak{B}$ of $SL(2, \mathbb{C})$ on X. A (G_a, G_m) pair is called *regular* when X^{G_a} is a single point $\{o\}$. Assuming (ϕ, λ) is regular, put

$$X_o = \{x \in X \mid \lim_{t\to\infty} \lambda(t) \cdot x = o\}.$$

Then X_o is a non empty G_m-stable affine open set, and the natural grading on $\mathbb{C}(X_o)$ is called the principal grading. Hence there exist coordinates $u_1, \ldots, u_n$ on X_o such that $\mathbb{C}(X_o) = \mathbb{C}[u_1, \ldots, u_n]$ where the u_i are homogeneous of positive degree with respect to the G_m-action on $\mathbb{C}(X_o)$.

In the above example, define a G_m-action λ on $\mathbb{P}^3$ by

$$\lambda(t) \cdot [z_0, z_1, z_2, z_3] = [t^3 z_0, t^1 z_1, t^{-1} z_2, t^{-3} z_3] = [z_0, t^{-2} z_1, t^{-4} z_2, t^{-6} z_3].$$

The action on the coordinates (u_1, u_2, u_3) is thus $\lambda(t) \cdot u = (t^{-2}u_1, t^{-4}u_2, t^{-6}u_3)$, so the coordinate functions thus have degrees 2,4 and 6 respectively. Notice that the components of V are also homogeneous of degrees 4,6 and 8, so $I(V)$ is a homogeneous ideal. Therefore, $\mathcal{A} = \mathbb{C}[u_1]/((u_1)^4)$ is a graded algebra with

$\deg(u_1) = 2$. This justifies $H^*(\mathbb{P}^3) \cong \mathcal{A}$. The final step which is the identification of u_1 and the element of cohomology associated to a hyperplane section will not be explained here.

Whenever X admits a regular (G_a, G_m) pair, X^{G_m} is finite and contains o [C95]. Moreover, $X_o = \{x \in X \mid \lim_{t\to\infty} \lambda(t) = o\}$ is an affine open cell in X and so there exist G_m-homogeneous coordinates $u_1, \ldots, u_n$ on X_o of positive degree. We now state the main result.

Theorem 10 (AC87,AC89)**.** *Let X be a smooth projective variety over $\mathbb{C}$ of dimension n admitting a regular (G_a, G_m)-action, and let V be the algebraic vector field on X generated by the G_a. Suppose $u_1, \ldots, u_n$ are homogeneous coordinates on X_o, and let $I(V) \subset \mathbb{C}[u_1, \ldots, u_n]$ be the ideal generated by $V(u_1), \ldots, V(u_n)$. Then for each i, $1 \le i \le n$, $V(u_i)$ is homogeneous of degree* $\deg(u_i) + 2$*, and there exists a graded ring isomorphism*

$$H^*(X) \cong \mathbb{C}[u_1, \ldots, u_n]/I(V).$$

Moreover, $V(u_1), \ldots, V(u_n)$ is a regular sequence, so

$$P(X,t) = \prod_{1\le i\le n} \frac{(1 - t^{\deg(V(u_i))})}{(1 - t^{\deg(u_i)})} = \prod_{1\le i\le n} \frac{(1 - t^{\deg(u_i)+2})}{(1 - t^{\deg(u_i)})},$$

Examples of varieties that admit a regular (G_a, G_m) pair include G/B, G/P for all parabolics P in G, Demazure varieties, smooth Schubert varieties and any smooth (G_a, G_m)- stable subvariety Y of a regular (G_a, G_m)-variety. Using this result in the case $GL(n, \mathbb{C})/B$, an interesting description of $H^*(X_w)$ for all Schubert varieties X_w using the Plucker relations [AAP92].

Surprisingly, if T is the maximal diagonal torus for the $\mathfrak{B}$-action associated to a regular (G_a, G_m), then the T-equivariant cohomology $H_T^*(X)$ has a nice description that under mild restrictions holds in the singular case too. We will briefly describe that here. Let V and W denote the holomorphic vector fields on X given by ϕ and λ respectively. Consider the holomorphic vector field on $X_o \times \mathbb{C}$ given by $Q(x,s) = V(x) + sW(x)$ and let $\mathcal{Z} = zero(Q)$. If $s \ne 0$, then all zeros of $Q(x,s)$ are simple. However, the zero at $(o, 0)$ is non-simple. In fact, $\mathcal{Z}$ is an affine curve in $X_o \times \mathbb{C}$ with $\chi(X)$ components which are obtained from the set of $\mathfrak{B}$-stable curves in $X \times \mathbb{P}^1$ by removing the points infinity. Then we have

Theorem 11 (BC04,CK10)**.** *The coordinate ring $\mathbb{C}(\mathcal{Z})$ defined above is isomorphic with $H_T^*(X)$. Furthermore, if $Y \subset X$ is a Zariski closed $\mathfrak{B}$-stable subvariety of X such that $H^*(X) \to H^*(Y)$ is surjective and $H^*(Y)$ is generated by Chern classes of $\mathfrak{B}$-equivariant vector bundles on Y, then we also have $H_T^*(Y) \cong \mathbb{C}(\mathcal{Z}_Y)$, where $\mathcal{Z}_Y$ denotes $\mathcal{Z} \cap (Y \times \mathbb{C})$. Furthermore, the obvious diagram commutes.*

A nice consequence of this result is a description of the equivariant cohomology of a Peterson variety. Finally, we remark that there is also a theory for (G_a, G_m)-varieties X where X^{G_a} isn't finite. (It is always connected, so X^{G_a} is either one point or infinite.) In [BO03], the cohomology algebras of certain stable map spaces was obtained in this way.

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