Betti numbers of unordered configuration spaces of a punctured torus

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(joint work with G. Cheong)

Introduction

For a topological space X and an integer $n \geq 0$, define the ordered configuration space as

$$F(X,n) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$

and the (unordered) configuration space as the topological quotient

$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

An important subject in topology is to study the structure of $\mathrm{Conf}^n(X)$. Various invariants, including the homotopy groups and the singular (co)homology groups, give rich information.

General Question

Given a smooth complex variety X, can we compute the i-th singular Betti number $h^i(\mathrm{Conf}^n(X)) := \dim_{\mathbb{Q}} H^i(\mathrm{Conf}^n(X); \mathbb{Q})$ for all integers i and n?

What do configuration spaces look like?

$$F(X,n) := \{(x_1,\dots,x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$
$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

First of all, $\mathrm{Conf}^0(X) = \{pt\}, \mathrm{Conf}^1(X) = X$.
For $n \geq 2$, it is helpful to consider the symmetric product

$$\mathrm{Sym}^n(X) := X^n/S_n = \{\text{multisets } \{x_1,\dots,x_n\} : x_i \in X\}.$$

Then $\mathrm{Conf}^n(X)$ is an open subspace of $\mathrm{Sym}^n(X)$ whose complement is the set of multisets with at least some repeated elements.
For $X = \mathbb{C}$, we have $\mathrm{Sym}^n(\mathbb{C}) \cong \mathbb{C}^n$ via

$$\underline{x} = \{x_1,\dots,x_n\} \mapsto (e_1(\underline{x}),\dots,e_n(\underline{x}))$$

where $e_i(\underline{x})$ is the i-th elementary symmetric polynomial.

What do configuration spaces look like?

$$\begin{array}{rcl} \mathrm{Sym}^n(\mathbb{C}) & \xrightarrow{\cong} & \mathbb{C}^n \\ \underline{x} = \{x_1, \ldots, x_n\} & \mapsto & (e_1(\underline{x}), \ldots, e_n(\underline{x})) \end{array}$$

Example $(\mathrm{Conf}^n(\mathbb{C}))$

1. $n = 2$. The isomorphism goes $\{a, b\} \mapsto (-(a+b), ab)$. The diagonal in $\mathrm{Sym}^2(\mathbb{C})$ is sent to the parabola $y = x^2/4$. So $\mathrm{Conf}^2(\mathbb{C})$ $\cong \mathbb{C}^2 - \{x\text{-axis}\} \cong \mathbb{C} \times \mathbb{C}^\times$. Betti numbers are $h^0 = h^1 = 1$.
2. $n \geq 2$. Then $\mathrm{Conf}^n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$ minus the the "discriminant hypersurface", i.e., the set of $(e_1, \ldots, e_n)$ such that $\mathrm{disc}(x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n) = 0$. If $n = 3$, the discriminant hypersurface is isomorphic to $\mathbb{C}^2$, but embedded in $\mathbb{C}^3$ in an interesting way!

History

$$F(X,n) := \{(x_1,\ldots,x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$

$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

Theorem (Arnol'd '69)

If $X = \mathbb{C}$, then $h^0(\mathrm{Conf}^n(X)) = 1$ for all $n \geq 0$, $h^1(\mathrm{Conf}^n(X)) = 1$ for all $n \geq 2$, and $h^i(\mathrm{Conf}^n(X)) = 0$ in all other cases.

Theorem (Kim '94, Vakil–Wood '15)

If X is $\mathbb{C}$ with $r \geq 0$ points removed, then

$$\sum_{i,n\geq 0} h^i(\mathrm{Conf}^n(X))(-u)^i t^n = \frac{1}{(1+ut)^r}\frac{1-ut^2}{1-t}$$

History

Theorem (Drummond-Cole–Knudsen '17)

If $X = \mathbb{P}^1 (\cong S^2)$, then $h^0(\mathrm{Conf}^n(X)) = 1 = h^2(\mathrm{Conf}^1(X))$ for $n \geq 0$, $h^3(\mathrm{Conf}^n(X)) = 1$ for $n \geq 3$, and $h^i(\mathrm{Conf}^n(X)) = 0$ in all other cases.

Theorem (Drummond-Cole–Knudsen '17)

If X is a smooth projective algebraic curve of genus g with r points removed, then there is an explicit formula for $h^i(\mathrm{Conf}^n(X))$ in terms of g, r, i, n involving binomial coefficients.

... but no "readable" and "comprehensible" pattern has been discovered.

Main Result

Theorem (Cheong–H.)

Let $X = E - P$, where E is an elliptic curve and P is a point of E. Then

$$\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}(-1)^i h^i(\mathrm{Conf}^n(X))u^{w(i)}t^n = \frac{(1-ut)^2(1-u^2t^2)}{(1-t)(1-u^3t^2)^2},$$

where

$$w(i) = \begin{cases} 3i/2, & i \text{ is even;} \\ (3i-1)/2, & i \text{ is odd.} \end{cases}$$

The novelty is the curious pattern of $w(i)$ that puts the Betti numbers in a rational function:

i	0	1	2	3	4	5	$\cdots$
$w(i)$	0	1	3	4	6	7	$\cdots$

Since $w(i)$ is strictly increasing, the above formula computes all $h^i(\mathrm{Conf}^n(X))$.

Main Result

To see how the formula works, compare

$$\frac{(1-ut)^2(1-u^2t^2)}{(1-t)(1-u^3t^2)^2} = 1 + (1-2u)t + (1-2u+2u^3)t^2 + (1-2u+4u^3-4u^4)t^3 + (1-2u+4u^3-5u^4+3u^6)t^4 + (1-2u+4u^3-5u^4+7u^6-6u^7)t^5 + \cdots$$

and the following table for $h^i(\mathrm{Conf}^n(E-P))$.

$n\backslash i$	0	1	2	3	4	5
0	1					
1	1	2				
2	1	2	2			
3	1	2	4	4		
4	1	2	4	5	3	
5	1	2	4	5	7	6

Main Ingredients of Proof: 1. Cut and Paste

We have combinatorial knowledge about configuration spaces in terms of how they are made up of simpler varieties via "cut and paste". Formally say $[X] = [U] + [Z] \in K_0(\mathrm{Var}_{\mathbb{C}})$ if U is an open subvariety of X and $Z = X \setminus U$. Here $K_0(\mathrm{Var}_{\mathbb{C}})$ is called the Grothendieck ring of complex varieties.

Theorem (Vakil–Wood '15)

For any variety X, we have

$$[K_X](t) = \frac{[Z_X](t)}{[Z_X](t^2)} \in K_0(\mathrm{Var}_{\mathbb{C}})[[t]],$$

where $[K_X](t) := \sum_{n=0}^{\infty} [\mathrm{Conf}^n(X)] t^n$ and $[Z_X](t) := \sum_{n=0}^{\infty} [\mathrm{Sym}^n(X)] t^n$.

For example, at degree 2, the formula is saying $[\mathrm{Conf}^2(X)] = [\mathrm{Sym}^2(X)] - [X]$.

Main Ingredients of Proof: 1. Cut and Paste

Theorem (Vakil–Wood '15)

(Summary) $[\mathrm{Conf}^n(X)]$ can be expressed as a polynomial in $[X], [\mathrm{Sym}^2(X)], \ldots, [\mathrm{Sym}^n(X)]$ via an easy formula.

Upshot

We are able to build unordered configuration spaces from symmetric spaces, whose cohomology is well-studied by Macdonald '75, Cheah '94.

Problem

Betti numbers do not interact well with cut-and-paste. In other words, $Z = X \setminus U$ does not imply $h^i(Z) = h^i(X) - h^i(U)$. We need to study a finer structure of singular cohomology groups.

Main Ingredients of Proof: 2. Mixed Hodge Theory

Recall the classical Hodge decomposition

$$H^i(X;\mathbb{C}) = \bigoplus_{\substack{p,q\geq 0 \\ p+q=i}} H^{p,q}(X).$$

Deligne develops a mixed Hodge theory for all complex varieties X, which gives vector spaces $H^{p,q;i}(X;\mathbb{Q})$ for *all* $p, q \geq 0$ (not just for $p+q=i$) whose dimensions (called the mixed Hodge numbers $h^{p,q;i}(X)$) satisfy

$$h^i(X) = \sum_{p,q\geq 0} h^{p,q;i}(X).$$

Given i, if there is an integer w such that $h^{p,q;i}(X) = 0$ unless $p+q=w$, then we say $H^i(X;\mathbb{Q})$ is pure of weight w. In this terminology, if X is smooth projective, then $H^i(X;\mathbb{Q})$ is pure of weight i for all i.

Main Ingredients of Proof: 2. Mixed Hodge Theory

Roughly speaking, the weight i part of $H^i(X)$ (i.e. $\bigoplus_{p+q=i} H^{p,q;i}(X)$) remembers the Hodge structure of a "smooth compactification" of X; the part of weight $> i$ is contributed by how far X is from being compact; the part of weight $< i$ comes from how far X is from being smooth.

Example

Let $X = \overline{X} - D$, where $\overline{X}$ is smooth projective and D is a smooth closed subvariety of codimension 1.

Then $H^i(X)$ only has parts of weight i and $i+1$, with

1. The weight i part is given by the image of $H^i(\overline{X}) \to H^i(X)$.
2. The weight $i+1$ part is a contribution of $H^{i-1}(D)$.

Note that both $\overline{X}$ and D has a Hodge decomposition in the classical sense. That's how the mixed Hodge structure of X is built.

Main Ingredients of Proof: 3. Purity

$H^i(X)$ is pure of weight w if $h^{p,q;i} = 0$ if $p + q \neq w$.

Purity is often the key property to go from the cut-and-paste combinatorics to the Betti numbers.

Fact

If X is smooth and $H^i(X)$ is pure of weight $w(i)$, then the polynomial

$$\sum_i (-1)^i h^i(X) u^{2n-w(i)} \in \mathbb{Z}[u]$$

only depends on $[X] \in K_0(\mathrm{Var}_{\mathbb{C}})$.

Reason behind: There is a mixed Hodge version of the Euler characteristic that only depends on $[X]$, given as an alternating sum of mixed Hodge numbers.

This polynomial is called the virtual Poincaré polynomial, denoted $P^{\mathrm{vir}}(X)$. It is easy to compute once we know $[X]$ well.

Motivating Example

Theorem (Kim '94)

Let X be $\mathbb{C}$ with $r \geq 0$ points removed. Then the ordered or unordered configuration spaces of X satisfy purity with weights $w(i) = 2i$.

Let $r = 0$. Using Vakil–Wood's result to compute $P^{\mathrm{vir}}(\mathrm{Conf}^n(\mathbb{C}))$, we will arrive at

$$\sum_{i,n} (-1)^i h^i(\mathrm{Conf}^n(\mathbb{C})) u^{2n-2i} t^n = \sum_n P^{\mathrm{vir}}(\mathrm{Conf}^n(\mathbb{C})) t^n$$
$$= \frac{1 - u^2t^2}{1 - u^2 t}$$
$$= 1 + u^2 t + (u^4 - u^2)t^2 + (u^6 - u^4)t^3 + (u^8 - u^6)t^4 + \dots$$

This implies Arnol'd's result: all the nonzero Betti numbers are

1. $h^0(\mathrm{Conf}^n(\mathbb{C})) = 1$ for all n, and
2. $h^1(\mathrm{Conf}^n(\mathbb{C})) = 1$ for $n \geq 2$.

Our Purity Result

The crux of our main theorem is the following purity statement:

Theorem (Cheong–H.)

Let X be an elliptic curve with one point removed. Then $H^i(\mathrm{Conf}^n(X);\mathbb{Q})$ is pure of weight $w(i)$ with

$$w(i) = \begin{cases} 3i/2, & i \text{ is even;} \\ (3i-1)/2, & i \text{ is odd.} \end{cases}$$

As a result,

$$\sum_{i,n} (-1)^i h^i(\mathrm{Conf}^n(X)) u^{2n-w(i)} t^n = \frac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2},$$

giving the main theorem. As another application of the purity, we get all mixed Hodge numbers of $\mathrm{Conf}^n(X)$.

Applications

Given our main theorem, the case of an r-punctured elliptic curve E_r can also be understood, using a nice formula (Napolitano '03) that expresses $H^i(\mathrm{Conf}^n(X-P))$ as a direct sum of $H^j(\mathrm{Conf}^m(X))$ if X is smooth *nonprojective* curve and $P \in X$.

In fact, the direct sum above can be upgraded to an isomorphism of mixed Hodge structures. Thus we get all $h^{p,q;i}(\mathrm{Conf}^n(E_r))$ as an application.

Theorem (H., preprint soon)

If X is a smooth *nonprojective* curve, and P is a point of X, then

$$h^{p,q;i}(\mathrm{Conf}^n(X-P)) = \sum_{t=0}^{\infty} h^{p-t,q-t;i-t}(\mathrm{Conf}^{n-t}(X))$$

I conjecture that an analogous version holds even in high dimension.

Break

In the rest of the talk, we will focus on the proof of purity of $\mathrm{Conf}^n(E - P)$.

Game plan

$$F(X,n) := \{(x_1,\ldots,x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$
$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

From now on, let X be a one-punctured elliptic curve. To understand the mixed Hodge structure of $\mathrm{Conf}^n(X)$, we divide into the following steps.

1. Explicitly describe a graded-commutative algebra $E_2(X,n)$ with generators and relations.
2. Explain how to read the mixed Hodge structure of $\mathrm{Conf}^n(X)$ from $E_2(X,n)$.
3. Work out the algebra and combinatorics of $E_2(X,n)$.

Outline of Proof: Step 1

Step 1

Explicitly describe a graded-commutative algebra $E_2(X, n)$ with generators and relations.

Recall that $H^*(V; \mathbb{C}) = \sum_{i=0}^{\infty} H^i(V; \mathbb{C})$ is a graded-commutative ring with respected to the cup product, for any topological space V. We first describe the cohomology ring of the Cartesian product X^n where $X = E - P$.

It turns out that

$$H^*(X^n; \mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1y_1, \ldots, x_ny_n)$$

where $\mathbb{C}[x_i, y_i]$ is the free graded-commutative algebra with generators x_i, y_i of degree one. Technically, $\mathbb{C}[x_i, y_i]$ can be constructed as the exterior algebra of the vector space $\bigoplus_{i=1}^{n} \mathbb{C}x_i \oplus \bigoplus_{i=1}^{n} \mathbb{C}y_i$.

Outline of Proof: Step 1

Step 1

Explicitly describe a graded-commutative algebra $E_2(X, n)$ with generators and relations.

The algebra $E_2(X, n)$ is defined as the graded-commutative algebra over $H^*(X^n; \mathbb{C})$ generated by formal variables g_{ij} for $1 \leq i \neq j \leq n$, subject to several explicit relations.

It actually has a bigrading: an element of $H^p(X^n)$ receives the bidegree $(p, 0)$, and g_{ij} has bidegree $(0, 1)$.

Thus we have

$$E_2(X, n) = \mathbb{C}[x_i, y_i, g_{ij}]/(\text{relations})$$

with $\text{bideg}(x_i) = \text{bideg}(y_i) = (1, 0)$ and $\text{bideg}(g_{ij}) = (0, 1)$

Outline of Proof: Step 2

Step 2

Explain how to read the mixed Hodge structure of $\mathrm{Conf}^n(X)$ from $E_2(X,n) = \mathbb{C}[x_i, y_i, g_{ij}]/(\text{relations})$.

The algebra $E_2(X,n)$ has a differential map $d : E_2(X,n) \to E_2(X,n)$ given by

1. $dx_i = dy_i = 0$
2. $dg_{ij} = -x_i y_j - x_j y_i$ (The right hand side here depends on specific geometry of X.)
3. The graded Leibniz rule.

Form a bigraded-commutative algebra $E_3(X,n)$ by taking the cohomology of d:

$$E_3(X,n) := \frac{\ker(d)}{\mathrm{im}(d)}$$

Outline of Proof: Step 2

Step 2

Explain how to read the mixed Hodge structure of $\mathrm{Conf}^n(X)$ from $E_2(X,n) = \mathbb{C}[x_i, y_i, g_{ij}]/(\text{relations})$.

We need another property special to X:

Important Observation

For $X = E - P$, we have $H^i(X)$ is pure of weight i. We remark that this property is not true for E minus two or more points.

By a theorem of Totaro '96 and the observation above, we have

Description

As graded-commutative algebras, $H^*(F(X,n)) \cong E_3(X,n)$. Moreover, the weight $p+2q$ part of $H^{p+q}(F(X,n))$ is the bidegree (p,q) component of $E_3(X,n)$.

Outline of Proof: Step 2

Step 2

Explain how to read the mixed Hodge structure of $\mathrm{Conf}^n(X)$ from $E_2(X,n) = \mathbb{C}[x_i, y_i, g_{ij}]/(\text{relations})$.

Taking S_n invariants on both sides of $H^*(F(X,n)) \cong E_3(X,n)$, we get $H^*(\mathrm{Conf}^n(X)) \cong E_3(X,n)^{S_n}$. We will show that

Key Proposition

$E_3(X,n)^{S_n}$ is concentrated at bidegrees (p,q) with $p-q=0$ or 1.

The purity theorem then follows, recalling that the bidegree (p,q) contributes to the weight $p+2q$ part of H^{p+q}. Intuitive explanation next page!

Outline of Proof: Step 2

$$H^*(\mathrm{Conf}^n(X)) \cong E_3(X,n)^{S_n}$$

$$p - q = 0 \text{ or } 1,\ i = p + q,\ w = p + 2q$$

Note how we read $w(i) = 0, 1, 3, 4, 6, 7, \ldots$ from the table.

q \ p	0	1	2	3	4	5
5	10	11	12	13	14	15
4	8	9	10	11	12	13
3	6	7	8	9	10	11
2	4	5	6	7	8	9
1	2	3	4	5	6	7
0	0	1	2	3	4	5

Weight distribution of $E_3(X,n)^{S_n}$ by bidegrees
Contribution to H^3 in blue

Outline of Proof: Step 3

Step 3

Work out the algebra and combinatorics of $E_2(X, n)$ and prove the key proposition.

Recall: $E_2(X, n) = \mathbb{C}[x_i, y_i, g_{ij}]/(\text{relations})$.
Some of the relations are $g_{ij} = g_{ji}$, $g_{ij}x_i = g_{ij}x_j$ and $g_{ij}y_i = g_{ij}y_j$, so we can introduct the notation

$$x_{ij} := g_{ij}x_i = g_{ij}x_j$$

$$y_{ij} := g_{ij}y_i = g_{ij}y_j$$

Recall: $E_3(X, n) = \left.\frac{\ker(d)}{\mathrm{im}(d)}\right|_{E_2(X,n)}$ and the goal is about $E_3(X, n)^{S_n}$.
Introduce the linear operator $e_n := \frac{1}{n!}\sum_{\sigma \in S_n} \sigma$. Since $V^{S_n} = e_n(V)$ for any S_n representation V over $\mathbb{C}$, we will use this operator to construct a generating set for $E_3(X, n)^{S_n}$.

Outline of Proof: Step 3

Lemma

$E_3(X,n)^{S_n}$ is generated by elements of the form

$$e_n(g^r_{i_1 i_2} x^{s_1}_j y^{s_2}_k x_{J_1} \dots x_{J_b} y_{K_1} \dots y_{K_c}) \mod \operatorname{im}(d)$$

where $|J_?| = |K_?| = 2$, all the lower indices are distinct, and $(r, s_1, s_2) \in \{0,1\}^3 - \{(1,0,0),(0,1,1)\}$.

Remarks.

1. The first step is to show that $E_2(X,n)^{S_n}$ is generated by elements of the same form, except that (r, s_1, s_2) ranges over $\{0,1\}^3$.
2. The differential of these generators are easy to describe because $dx_{ij} = dy_{ij} = 0$. No longer true if X were projective!

One can check that all these generators have bidegree (p,q) with $p - q = 0$ or 1, by directly checking the six choices of (r, s_1, s_2). This finishes the proof of everything.

Remarks about Results

1. Unlike in the case of (punctured) affine line, the *ordered* configuration spaces of X has no purity. (Taking S_n invariants is required.)
2. If X is a punctured curve of genus at least 2, or an elliptic curve with at least 2 punctures, then there is no purity result for $\mathrm{Conf}^n(X)$.
3. The same proof shows the purity for the Galois module $H^*(\mathrm{Conf}^n(X); \mathbb{Q}_\ell)$, where X is a one-punctured elliptic curve over a finite field.
4. We wonder if anything can be said about $H^i(\mathrm{Conf}^n(X); \mathbb{Z})$. They are known to have torsion.

Further Work

1. I expect that the method works for a one-punctured smooth projective variety X in a way nicer than general, because the top cohomology of X vanishes (for being nonprojective) while the mixed Hodge structure of X is still pure. The next case to try is higher genus curves.
2. Again, one can understand the multi-punctured case using the one-punctured case if the following conjecture holds:

Conjecture (H.) and Theorem (H.) if $\dim X = 1$

If X is a smooth *nonprojective* variety, and P is a point of X, then

$$h^{p,q;i}(\mathrm{Conf}^n(X-P)) = \sum_{t=0}^{\infty} h^{p-dt,q-dt;i-kt}(\mathrm{Conf}^{n-t}(X)),$$

where $d = \dim_{\mathbb{C}} X$ and $k = 2d-1$.

Summary

1. We found a coherent way to understand all the Betti numbers of configuration spaces of a one-punctured torus at once. Thus also for multi-punctured torus.
2. The cut-and-paste of configuration spaces are well-known, and the mixed Hodge theory can take advantage of it.
3. The story for nonprojective varieties are somehow very different from the projective case because nonprojective varieties have more vanishings in the cohomology.

For the case of smooth *projective* curves, see R. Pagaria's "Coh. of conf. spaces of surfaces" where mixed Hodge numbers are computed.

Thank you!

For more details, please refer to

1. G. Cheong and Y. Huang, *Rationality for the Betti numbers of the unordered configuration spaces of a punctured torus*, arXiv:2009.07976