Cohomology of configuration spaces of punctured varieties

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Introduction

For a topological space X and an integer $n \geq 0$, define the ordered configuration space as

$$F(X,n) := \{(x_1,\ldots,x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$$

and the (unordered) configuration space as the topological quotient

$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

An important subject in topology is to study the structure of $\mathrm{Conf}^n(X)$. Various invariants, including the homotopy groups and the singular (co)homology groups, give rich information.

In this talk, we focus on the cohomology groups.

What do configuration spaces look like?

$$F(X,n) := \{(x_1,\dots,x_n)\in X^n : x_i \neq x_j \text{ for } i\neq j\}$$
$$\mathrm{Conf}^n(X) := F(X,n)/S_n.$$

First of all, $\mathrm{Conf}^0(X)=\{pt\}, \mathrm{Conf}^1(X)=X$.
For $n\geq 2$, it is in general hard to visualize $\mathrm{Conf}^n(X)$ due to the quotient, but if $X=\mathbb{R}^2=\mathbb{C}$, we can use $\mathbb{C}^n/S_n\cong\mathbb{C}^n$ and get

1. $\mathrm{Conf}^2(\mathbb{C})\cong\mathbb{C}\times\mathbb{C}^\times$, homotopy equivalent to S^1. (elementary)
2. $\mathrm{Conf}^3(\mathbb{C})$ is homotopy equivalent to S^3 minus a trefoil knot.
3. $\mathrm{Conf}^n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$ minus the "discriminant hypersurface" cut out by one polynomial equation. (complicated for $n\geq 3$)

On the other hand, their singular cohomology has a much simpler answer.

Theorem (Arnol'd '69)

$H^0(\mathrm{Conf}^n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ *for all* $n\geq 0$, $H^1(\mathrm{Conf}^n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ *for all* $n\geq 2$, *and* $H^i(\mathrm{Conf}^n(\mathbb{C}),\mathbb{Q})=0$ *for all other choice of* i *and* n.

A splitting phenomenon

It appears that for many spaces X, puncturing has a consistent effect on the cohomology groups of its configuration spaces. Here, $X - P$ always means X with an arbitrary point $P \in X$ removed.

Theorem (Goryunov '78, $X = \mathbb{R}^2$)

$$H^i(\mathrm{Conf}^n(\mathbb{R}^2 - P), \mathbb{Z}) \cong \bigoplus_{t=0}^{\infty} H^{i-t}(\mathrm{Conf}^{n-t}(\mathbb{R}^2), \mathbb{Z})$$

Theorem (?, $X = \mathbb{R}^d$)

$$H^i(\mathrm{Conf}^n(\mathbb{R}^d - P), \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} H^{i-(d-1)t}(\mathrm{Conf}^{n-t}(\mathbb{R}^d), \mathbb{Q})$$

Theorem (Fuchs '74, X a plane with punctures)

A similar "stable" formula, i.e. valid for Conf^∞, or Conf^n with $n \gg 0$.

A splitting phenomenon

Theorem (Napolitano '03, X basically any nonclosed surface)

Let M be a surface (not necessarily orientable, possibly with boundary), and O, P be two interior points of M. For $X = M - O$, we have

$$H^i(\mathrm{Conf}^n(X - P), \mathbb{Z}) \cong \bigoplus_{t=0}^{\infty} H^{i-t}(\mathrm{Conf}^{n-t}(X), \mathbb{Z})$$

Theorem (Kallel '08, X a punctured manifold)

Let M be an oriented closed connected manifold of even dimension, and $P, P_1, P_2, \ldots, P_r$ $(r \geq 1)$ be points of M. For $X = M - \{P_1, \ldots, P_r\}$,

$$H^i(\mathrm{Conf}^n(X - P), \mathbb{F}) \cong \bigoplus_{t=0}^{\infty} H^{i-(d-1)t}(\mathrm{Conf}^{n-t}(X), \mathbb{F})$$

for any field $\mathbb{F}$.

A splitting phenomenon

In this talk, we say that a manifold X of dimension d satisfies the splitting phenomenon (in $\mathbb{Q}$ coefficients) if

$$H^i(\mathrm{Conf}^n(X-P),\mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} H^{i-(d-1)t}(\mathrm{Conf}^{n-t}(X),\mathbb{Q})$$

If X is a closed manifold, it is known that the splitting phenoemenon for X does not hold. In fact, a (completely different) relation holds in $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Question

What other connected nonclosed manifold satisfies the splitting phenomenon? Does the splitting phenomenon hold in a sense stronger than the Betti numbers (dimensions of rational cohomology)?

From now on, we focus on the nice case where X is a smooth complex variety.

Main result

Theorem (H., to be continued)

Let X be a (connected) noncompact smooth (complex) variety of complex dimension d in one of the three cases:

1. an r-punctured smooth projective variety, with $r \geq 1$;
2. an r-punctured affine plane $\mathbb{C}^d$, with $r \geq 0$;
3. $(\mathbb{P}^2 - C) - \{r \geq 0 \text{ points}\}$, where C is a smooth plane curve.

Then there are explicit isomorphisms that remember additional structures:

$$H^i(\mathrm{Conf}^n(X - P), \mathbb{Q}) \xrightarrow{\cong} \bigoplus_{t=0}^{\infty} H^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(X), \mathbb{Q}), \tag{1}$$

$$H^i(F(X - P, n), \mathbb{Q}) \xrightarrow{\cong} \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(X, n-t), \mathbb{Q}), \tag{2}$$

where Ind means the induction of a group representation.

Main result

Remarks.

1. If X is in the first two cases above, then (1) is the $\mathbb{Q}$-coefficient version of known results. But we will see that the new proof keeps track of "mixed Hodge numbers", an invariant finer than Betti numbers. Moreover, the isomorphisms constructed here interact with the cup product well.
2. The actual condition for X here is a technical condition in terms of mixed Hodge theory, which is satisfied by all three cases above as well as many other examples.
3. The strongest statement one can hope for is

Conjecture. For any noncompact smooth d-dimensional variety X, we have

$$H^i(F(X-P,n),\mathbb{Z}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(X,n-t),\mathbb{Z})(-dt)$$

as mixed Hodge structures, where $(-dt)$ denotes a "Tate twist" that only affects the mixed Hodge structure.

Technical overview

1. The main tool is a Leray-type spectral sequence, with an important page that can be explicitly described as a differential graded algebra (dga) similar to the work of Cohen, Taylor, Kriz and Totaro.
2. The key is an isomorphism involving the dga above, constructed explicitly and "artificially". This is the main novelty and the unexpected part of my method.
3. The mixed Hodge theory ensures degeneracy of the spectral sequence. This is the reason why we work on complex varieties and impose conditions on X.

Mixed Hodge structures

A pure Hodge structure of weight m is an abelian group H with a decomposition of $\mathbb{C}$-vector spaces $H \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$ such that $H^{p,q}$ and $H^{q,p}$ are complex conjugates. The classical example of a pure Hodge structure of weight m is $H^m(X, \mathbb{Z})$ for a smooth projective variety X.

Deligne generalized the Hodge theory to any complex variety X, except that now $H^m(X, \mathbb{Z})$ "does not only have weight-m part". A mixed Hodge structure is an abelian group H equipped with

1. an increasing weight filtration $0 = \cdots \subseteq W_{m-1} \subseteq W_m \subseteq \cdots = H_{\mathbb{Q}}$;
2. a decreasing Hodge filtration $H_{\mathbb{C}} = \cdots \supseteq F_{p-1} \supseteq F_p \supseteq \cdots = 0$,

such that the weight-m piece $\mathrm{Gr}_m^W H_{\mathbb{Q}} := W_m / W_{m-1}$ is a pure HS of weight m determined by the Hodge filtration in a certain way.

One may view a MHS as a weight filtration on $H_{\mathbb{Q}}$ together with the pure HS $\mathrm{Gr}_m^W H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$ for each weight-graded piece, but with some additional data encoded in the Hodge filtration.

Mixed Hodge structures

Deligne constructed a mixed Hodge structure on $H^m(X,\mathbb{Z})$ for any complex variety X with $W_{-1}=0$ and $W_{2m}=H^m(X,\mathbb{Q})$, so there are only pieces of weight in $[0,2m]$.
Roughly speaking, the weight-m piece corresponds to a "smooth compact model" of X, pieces of weight $> m$ account for the noncompactness of X (e.g., its complement, if X is open in some compact $\overline{X}$), and pieces of weight $< m$ account for the singularity of X (e.g., the exceptional divisor from blowup).
Takeaway: if X is smooth, then $H^m(X,\mathbb{Q})$ only has pieces of weight in $[m,2m]$. If X is compact, then the weights are concentrated in $[0,m]$.

Mixed Hodge structures form an abelian category, so we can talk about morphisms, extensions and spectral sequences. For example, a rational MHS $H_{\mathbb{Q}}$ is an iterated extension of its weight graded pieces $\mathrm{Gr}_m^W H_{\mathbb{Q}}$, but the HS of them do not determine the MHS $H_{\mathbb{Q}}$.

Main result in MHS

We can now finish stating the main result after setting up some language to compare mixed Hodge structures.

Tate twist. Given a MHS H and $n \in \mathbb{Z}$, the Tate twist $H(-n)$ is a MHS with underlying group H but with the filtrations shifted so that $H(-n)^{p,q} = H^{p-n,q-n}$.

Equivalence. Given two rational MHS H and H', there are several notions weaker than isomorphism, listed in ascending order of strength:

1. having the same mixed Hodge numbers, i.e., $\dim_{\mathbb{C}} H^{p,q} = \dim_{\mathbb{C}} H'^{p,q}$.
2. equivalence up to extension problem, so $[H] = [H'] \in K_0(\mathrm{MHS})$.
3. having isomorphic weight-graded pieces, i.e., $\mathrm{Gr}_m^W(H) \cong \mathrm{Gr}_m^W(H')$ as pure HS.

It is enough to care about (1) for most purposes, but in the three cases in the main result, we are able to get (3). Write $H \sim H'$ if they are equivalent in the sense (3).

Main result in MHS

Theorem (H., continued)

Let X be as before. Then as mixed Hodge structures, we have

$$H^i(\mathrm{Conf}^n(X-P),\mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} H^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(X),\mathbb{Q})(-dt), \tag{3}$$

$$H^i(F(X-P,n),\mathbb{Q}) \sim \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(X,n-t),\mathbb{Q})(-dt). \tag{4}$$

Remark

Note that (4) is an upgrade of (3). Equation (4) implies (3) by taking S_n invariants on both sides. The right hand side needs the fact: $(\mathrm{Ind}_{S_{n-t}}^{S_n} V)^{S_n} \cong V^{S_{n-t}}$ for any S_{n-t} representation V.

Applications

Given the main result, we are able to compute the mixed Hodge numbers of configuration spaces of a punctured variety, as long as we know those of a one-punctured variety, which I expect to have a nicer answer. My recent joint work computes those of a one-punctured elliptic curve, which leads to the case of multi-punctured elliptic curves using the splitting result.

Theorem (Cheong, H.)

$$\sum_{p,q,i,n\geq 0}(-1)^i h^{n-p,n-q;i}(\mathrm{Conf}^n(E-P))x^p y^q u^{2n-w(i)}t^n = \frac{(1-xut)(1-yut)(1-xyu^2t^2)}{(1-xyu^2t)(1-xut^2)(1-yut^2)},$$

where

$$w(i)=\begin{cases}3i/2, & i \text{ is even;}\\ (3i-1)/2, & i \text{ is odd.}\end{cases}$$

A spectral sequence

Observe that $F(X,n) = X^n - \bigcup_{1\le i\ne j\le n} \Delta_{ij}$, where $\Delta_{ij} = \{x_i = x_j\}$ is a big diagonal. The cohomology of a space minus a union of closed subspaces can often be computed via a spectral sequence. A prototype appears in Deligne's work, and versions of this spectral sequence appear in various forms, generalities and strengths.

Most of its variants (e.g. Totaro, Petersen) can be recognized by the following features.

1. Goresky–MacPherson's computation of $H^*(\mathbb{R}^d - \bigcup_i A_i, \mathbb{Z})$, the complement of affine subspaces, in terms of the combinatorial data of their intersection. In specific, a stratum is the intersection of several subspaces, and what matters is the partially ordered set of all strata ordered by inclusion.
2. A spectral sequence converging to the desired cohomology, with a page described in terms of the cohomology of each stratum and the poset of strata.

A spectral sequence

For the arrangements of closed subspaces we will use today, the poset of strata is of a special kind called "geometric lattice". The contribution of such a poset in the spectral sequence has a simpler description given by the "Orlik–Solomon algebra", as will be made clear below.

Theorem (Totaro '96)

Let M be an orientable manifold of dimension $2d$. Then there is a spectral sequence $E_{2d}^{p,q} \implies H^{p+q}(F(M,n),\mathbb{Z})$ such that $E_{2d} := \bigoplus_{p,q} E_{2d}^{p,q}$ is a dga over $H^*(M^n,\mathbb{Z})$ generated by generators g_{ij} of bidegree $(0, 2d-1)$ (one for each big diagonal Δ_{ij}) subject to relations

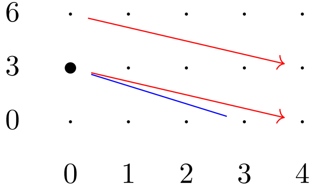
1. $g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} = 0$
2. $g_{ij}A = 0$ *if* $A \in H^*(M^n,\mathbb{Z})$ *and* $A|_{\Delta_{ij}} = 0$.

The page-$2d$ differential is given by $\mathrm{d}_{2d} g_{ij} = [\Delta_{ij}] \in H^{2d}(M^n,\mathbb{Z})$.

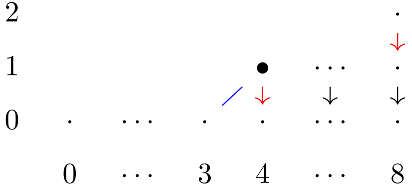
The relation (1) is a relation in the OS algebra coming from the dependence among big diagonals: $\Delta_{ij} \supseteq \Delta_{jk} \cap \Delta_{ki}$.

A spectral sequence

If we instead apply a more recent spectral sequence (Tosteson, or dual of Petersen) to the same arrangement, we get the same SS but in different page and tilted: we get $E_1^{p,q} \implies H^{p-q}(F(M,n),\mathbb{Z})$; the dga E_1 is generated in the same way, except that g_{ij} is of bidegree $(2d,1)$.



Totaro: E_{2d} with d_{2d}



Tosteson: E_1 with d_1

Comparing spectral sequences for $d = 2$. Dots are the bigraded pieces that can be nonzero. **Bullets** are where g_{ij} lives. Arrows are differentials. Red arrows correspond. Terms on the blue lines contribute to $H^3(F(M,n),\mathbb{Z})$.

For convenience, we will use the E_1 spectral sequence.

A spectral sequence

Now let X be a smooth complex variety of dimension d, and $P^1, \dots, P^r$ $(r \geq 0)$ be points of X. Write $X_r = X - \{P^1, \dots, P^r\}$, then $F(X_r, n) = X^n - \bigcup \Delta_{ij} - \bigcup \Delta_i^s$, where $\Delta_i^s = \{x_i = P^s\}$. The same recipe can be applied to this arrangement as well, because its poset of strata is also a "geometric lattice".

Lemma. There is a spectral sequence $E_1^{p,q} \implies H^{p-q}(F(X_r, n), \mathbb{Z})$ such that E_1 is a dga over $H^*(X^n, \mathbb{Z})$ generated by g_{ij} and g_i^s of bidegree $(2d, 1)$ (corresponding to Δ_{ij} and Δ_i^s) subject to

1. $g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} = g_{ij}g_j^s + g_j^s g_i^s + g_i^s g_{ij} = 0$
2. $g_{ij}A = 0$ if $A|_{\Delta_{ij}} = 0$ and $g_i^s A = 0$ if $A|_{\Delta_i^s} = 0$.
3. $g_i^s g_i^t = 0$ if $s \neq t$.

The differential is given by $\mathrm{d}_1 g_{ij} = [\Delta_{ij}]$ and $\mathrm{d}_1 g_i^s = [\Delta_i^s]$.

The second part of (1) is the dependence $\Delta_i^s \supseteq \Delta_j^s \cap \Delta_{ij}$. Relation (3) is due to $\Delta_i^s \cap \Delta_i^t = \varnothing$, another rule in OS algebra.

A spectral sequence

Lemma. There is a spectral sequence $E_1^{p,q} \implies H^{p-q}(F(X_r, n), \mathbb{Z})$ such that E_1 is a dga over $H^*(X^n, \mathbb{Z})$ generated by g_{ij} and g_i^s of bidegree $(2d, 1)$ (corresponding to Δ_{ij} and Δ_i^s) subject to certain relations. The differential is given by $\mathrm{d}_1 g_{ij} = [\Delta_{ij}]$ and $\mathrm{d}_1 g_i^s = [\Delta_i^s]$.

This lemma is not written in the literature. I explain some claims further:

1. There could be more relations coming from OS algebra, but a careful elementary argument shows that the relations here suffice.
2. A direct application of Tosteson shows that $E_1^{p,q}$ is as claimed, as an abelian group.
3. The dga structure comes from identification with the E_{2d} page of the Totaro's Leray SS.
4. This is a spectral sequence of MHS, if we assign Hodge type (d, d) to the generators g_{ij} and g_i^s (technically a Tate twist by $(-qd)$ on $E_1^{p,q}$). This is because the SS of Petersen respects MHS.

Key isomorphism

Write $[n] = \{1, \ldots, n\}$ and $[n] - i = \{1, \ldots, n\} \setminus \{i\}$. Use the notation $F(X_r, I)$ for any finite index set I, which means the coordinates are labeled by elements of I. Denote by $E_1(X_r, I)$ the dga E_1 for $F(X_r, I)$.

Lemma (H.)

We have an isomorphism of bigraded $\mathbb{Q}$-vector spaces

$$\Phi : E_1(X_{r-1}, n)_{\mathbb{Q}} \oplus \bigoplus_{i=1}^{n} E_1(X_r, [n] - i)_{\mathbb{Q}} \to E_1(X_r, n)_{\mathbb{Q}}$$

such that $\Phi|_{E_1(X_{r-1},n)_{\mathbb{Q}}}$ is the natural map, and $\Phi|_{E_1(X_r,[n]-i)_{\mathbb{Q}}}$ is the $E_1(X_{r-1}, [n] - i)_{\mathbb{Q}}$-module map that sends 1 to g_i^r and sends g_j^r to $g_{ij} g_i^r$ for all $j \in [n] - i$.

Moreover, the lemma holds in $\mathbb{Z}$ coefficients if Künneth's formula works: $H^*(X^n, \mathbb{Z}) \cong H^*(X, \mathbb{Z})^{\otimes n}$ (true if $H^*(X, \mathbb{Z})$ is torsion free).

Key isomorphism

We remark that the isomorphism is constructed artificially by working on the generators and relations. It is not known whether it can be constructed from functorial or topological maps.

The isomorphism on E_1 can be used to compare $H^*(X_r)$ and $H^*(X_{r-1})$, eventually leading to the splitting phenomenon for X_{r-1}. But to go from E_1 to H^*, one needs to

1. take the cohomology (kernel mod image) of the differential maps to compute the next page;
2. wait for $E_h^{p,q}$ to stabilize, getting $E_\infty^{p,q} = E_h^{p,q}, h \gg 0$.
3. Then H^i is an iterated extension of $E_\infty^{p,q}$, $p - q = i$.

The difficulty is that the differential maps on E_2 or later pages are beyond control.

Why noncompact?

We claim that Φ commutes with the first-page differential d_1 when X is not compact.

If X is noncompact, the top cohomology $H^{2d}(X, \mathbb{Z})$ vanishes. Recall that the artificial part of Φ sends 1 to g_i^r and g_j^r to $g_{ij} g_i^r$. By the graded Leibniz rule, it suffices to show that the differentials of all these elements vanish.

Using the rule of d_1, one can show $\mathrm{d}_1 g_i^r$ is the pullback of a class in $H^{2d}(X)$, which must vanish. On the other hand, $\mathrm{d}_1(g_{ij} g_i^r) = (\mathrm{d}_1 g_{ij}) g_i^r$ by Leibniz, and we can again show that $(\mathrm{d}_1 g_{ij})|_{\Delta_i^r}$ is a pullback from $H^{2d}(X)$ and vanishes. But a relation describing E_1 says $g_i^r A = 0$ for $A|_{\Delta_i^r} = 0$.

Upshot. The isomorphism Φ descends to the E_2 page.

Degeneracy

We hope the isomorphism Φ descends to the E_∞ page. The problem is solved if $E_2 = E_\infty$, i.e., if all differential maps on E_2 or later pages vanish. We say the spectral sequence degenerates at the E_2 page.

Lemma

Assume that there is a rational number $1 \le w \le 2$ such that $H^i(X, \mathbb{Q})$ is pure of weight wi for all i. (If wi is not an integer, this forces $H^i(X, \mathbb{Q}) = 0$.) Then for any $r, n \ge 0$, the spectral sequence $E_1^{p,q}(X_r, n)_{\mathbb{Q}} \implies H^{p-q}(F(X_r, n), \mathbb{Q})$ degenerates at the E_2 page.

Proof.

If $w = 1$, then the piece $E_1^{p,q}$ is pure of weight p. The same holds for later pages because they are subquotients of E_1. So every differential (except d_1) connects pieces of different weights, so it must vanish because the SS is in the category of MHS. It is a classical argument of Deligne and Totaro. If $w > 1$, we keep track of the weights of pieces and argue the same. □

Degeneracy

Lemma

Assume that there is a rational number $1 \leq w \leq 2$ such that $H^i(X, \mathbb{Q})$ is pure of weight wi for all i. (If wi is not an integer, this forces $H^i(X, \mathbb{Q}) = 0$.) Then for any $r, n \geq 0$, the spectral sequence $E_1^{p,q}(X_r, n)_{\mathbb{Q}} \implies H^{p-q}(F(X_r, n), \mathbb{Q})$ degenerates at the E_2 page.

The assumption holds for the following examples of X.

1. X is a one-punctured smooth projective variety. In this case $w = 1$. The reason is that puncturing once only kills the top cohomology of a smooth projective variety and changes nothing else.
2. $X = \mathbb{C}^d$. In this case w is arbitrary because $H^i(X, \mathbb{Z}) = 0$ for $i > 0$.
3. $X = \mathbb{P}^2 - C$, where C is smooth plane curve of genus g. Then $w = 3/2$. In fact, the only nonzero rational cohomologies are $H^0(X, \mathbb{Q}) = \mathbb{Q}$ and $H^2(X, \mathbb{Q}) = \mathbb{Q}^{2g}$ (pure of weight 3).

Completion of the proof

For X in the cases above, we get an isomorphism of rational MHS

$$\Phi_\infty^{p,q} : E_\infty^{p,q}(X_{r-1}, n)_\mathbb{Q} \oplus \bigoplus_{i=1}^{n} E_\infty^{p-2d,q-1}(X_r, [n]-i)_\mathbb{Q}(-d) \to E_\infty^{p,q}(X_r, n)_\mathbb{Q},$$

where the shift and twist accounts for the fact that g_{ij} and g_i^s have bidegree $(2d, 1)$ and Hodge type (d, d). (Recall $\Phi : 1 \mapsto g_i^r, g_j^r \mapsto g_{ij} g_i^r$.)
Fix k. Take an iterated extension of the equation for all p, q with $p - q = k$, we get an equivalence of MHS up to extension problem

$$H^k(F(X_{r-1}, n), \mathbb{Q}) \oplus \bigoplus_{i=1}^{n} H^{k-(2d-1)}(F(X_r, [n]-i), \mathbb{Q})(-d) \\ \sim H^k(F(X_r, n), \mathbb{Q})$$

We remark that the iterated extension involves pieces of distinct weights, so we get the stronger sense of "$\sim$" equivalence discussed before.

Completion of the proof

We wrap up the proof with a representation-theoretic observation.

Write $V(I) = H^*(F(X_r, I), \mathbb{Q})$ for an index set I, noting that it has an action by the permutation group of I. Then as S_n representations,

$$\bigoplus_{i=1}^{n} V([n]-i) \cong \mathrm{Ind}_{S_{n-1}}^{S_n} V([n-1])$$

This gives (omitting coefficients $\mathbb{Q}$ in the notation)

$$H^k(F(X_r,n)) \sim H^k(F(X_{r-1},n)) \oplus \mathrm{Ind}_{S_{n-1}}^{S_n} H^{k-(2d-1)}(F(X_r,n-1))(-d).$$

The second part of the splitting theorem for X_{r-1} (recapped below) follows from induction. QED.

$$H^k(F(X_r,n)) \sim \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{k-(2d-1)t}(F(X_{r-1},n-t))(-dt)$$

Summary of what's new

Main takeaway. We investigated the "splitting phenomenon" relating the cohomology of $\mathrm{Conf}^n(X-P)$ and $\mathrm{Conf}^m(X)$ for noncompact X.

1. If X is a punctured smooth projective variety or affine space, we upgrade known results to MH numbers.
2. For X as above, the isomorphism is explicit and interacts well with cup product, thanks to an observation by Deligne: $H^*(F(X_r, n), \mathbb{Q}) \cong E_2(X, n)_{\mathbb{Q}}$ as rings (not necessarily as MHS).
3. Even for Betti numbers, the case of $\mathbb{P}^2 - C$ may be new.
4. We proved a splitting result for ordered configuration spaces.
5. This machinery also works in the category of Galois representations of ℓ-adic cohomology. We get a splitting theorem in that sense, where the isomorphism is up to semisimplification.

We note that (2)(3)(4) are of topological interest even if we don't care about MHS, algebraic geometry or arithmetics.

Further work

Conjecture. For any noncompact smooth d-dimensional variety X, we have

$$H^i(F(X-P,n),\mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(X,n-t),\mathbb{Q})(-dt)$$

To extend the main result to more examples of noncompact X, there are two independent directions:

1. Try to control the pages after E_2, either by looking for degeneracy without the weight argument, or by extending Φ to a "morphism of SS". The latter is usually achieved by the functoriality of the SS in question, but as a warning, any construction must make use of noncompactness of X.
2. Study the spectral sequence for $F(X-\bigcup Z_i,n)$ where X is a one-punctured smooth projective variety and Z_i's are closed subvarieties of X (not necessarily points). Then we have degeneracy, but the poset of strata is more complicated.

Remarks about integer coefficients

One can attempt cases of the conjecture above in $\mathbb{Z}$ coefficients, because in some cases, the splitting is known in $\mathbb{Z}$ coefficients (e.g., Napolitano).

My method has several places that require rational coefficients, and one may need to deal with these problems when working with integer coefficients.

1. Künneth's formula may not hold in $\mathbb{Z}$ coefficients.
2. The weight filtration is defined on $H_{\mathbb{Q}}$ for a MHS H, so the degeneracy we can achieve so far is in $\mathbb{Q}$ coefficients.
3. The integral cohomology groups are not determined by the E_∞ page of the SS because of the extension problem.
4. The formula $H^*(\mathrm{Conf}^n(X)) = H^*(F(X,n))^{S_n}$ only holds in rational coefficients.

Advertisement

I will talk about a heavily AG motivated combinatorics question at 4-5pm, Tuesday Nov. 24 in UBC Discrete Mathematics Seminar. It is partially motivated by the work of Kai Behrend and Jim Bryan.

Thank you!

For more details, please refer to

1. Y. Huang, *Cohomology of configuration spaces on punctured varieties*, preprint upon request.