Zeta functions on orders

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Dedekind zeta function

- R: a Dedekind domain (say, the ring of integers in a number field or a function field).
- $\zeta_R(s) := \sum_I |R/I|^{-s}$, summed over ideals I with $|R/I| < \infty$.
- Has meromorphic continuation and functional equation $s \mapsto 1 - s$.

A generalization

Lattice zeta function

- Let M be an R-module.
- $\zeta_M(s) := \sum_L |M/L|^{-s}$, summed over R-submodules L with $|M/L| < \infty$.
- When $M = R$, this is just the Dedekind zeta function.
- (Solomon '77) When R is Dedekind and $M = R^d$, then $\zeta_M(s) = \zeta_R(s)\zeta_R(s-1)\ldots\zeta_R(s-d+1)$. From this, it is easy to verify a functional equation $s \mapsto d - s$.

An analog

Cohen–Lenstra zeta function

- $\widehat{\zeta}_R(s) := \sum_M |M|^{-s}/|\mathrm{Aut}(M)|$, summed over finite R-modules M up to isomorphism.
- (Cohen–Lenstra '83) When R is Dedekind, then $\widehat{\zeta}_R(s) = \zeta_R(s+1)\zeta_R(s+2)\ldots$. Moreover, it has meromorphic continuation and a functional equation $s \mapsto 1-s$.

Question

What if R is a non-maximal order, thus not Dedekind?

Insight

Function-field case gives us lots of interesting examples. They are also geometric since order = singularity curve.

Dedekind zeta function for orders

Let R be an order in the maximal order $\widetilde{R}$.

Facts

- $\zeta_R(s)$ has meromorphic continuation; in fact, $\zeta_R(s)/\zeta_{\widetilde{R}}(s)$ is entire.
- When R is "Gorenstein", then $\zeta_R(s)$ has a functional equation $s \mapsto 1 - s$.
- When R is the coordinate ring of a singular curve X over $\mathbb{F}_q$, $\zeta_R(s)$ encodes point-counts of Hilbert schemes of points on X.
- If X has only planar singularities, then R is Gorenstein.
- Typical example: $R = \mathbb{F}_q[X, Y]/(Y^m - X^n) \subseteq \widetilde{R} = \mathbb{F}_q[[T]]$, m, n pairwise coprime, via $X = T^m, Y = T^n$.
- The exact formula here involves a combinatorially interesting polynomial in $q, t = q^{-s}$ (called the generalized q, t-Catalan number). (Gorsky–Mazin '13).
- (Oblomkov–Rasmussen–Shende conjecture '18) For any curve with planar singularities, $\zeta_R(s)$ should encode knot invariants!!!

Lattice zeta function for orders

Let R be an order and M be an "R-lattice of rank d" (e.g., R^d or its finite-index submodule).

New results (and background)

- $\zeta_M(s)$ has meromorphic continuation; in fact, $\zeta_R(s)/\zeta_{\widetilde{R}^d}(s)$ is entire. (Bushnell–Reiner '81)
- (H.–Jiang) When R is Gorenstein, then $\zeta_{R^d}(s)$ has a functional equation $s \mapsto d - s$.
- (H.–Jiang) When $R = \mathbb{F}_q[X,Y]/(Y^2 - X^n)$, $n \geq 2$, q odd, then $\zeta_{R^d}(s)$ is an explicit polynomial in $q, t = q^{-s}$ involving partitions, Hall polynomials and q-hypergeometric functions.
- Thus: the functional equation implies combinatorial identities; we have found a nontrivial direct proof for the case $n = 2$ or n odd.

Cohen–Lenstra zeta function for orders

Let R be an order. Consider $\widehat{\zeta}_R(s)$

- Nothing is known in the number field case.
- Assume function field from now. Then $\widehat{\zeta}_R(s)$ is related to counting tuples of pairwise commuting matrices satisfying Diophantine equations. E.g. $AB = BA = 0$ for $R = \mathbb{F}_q[X,Y]/(XY)$. (H.)
- (H.–Jiang) $\widehat{\zeta}_R(s)$ can be computed in terms of $\zeta_{R^d}(t)$ for all $d \geq 0$.
- (H.–Jiang) When $R = \mathbb{F}_q[X,Y]/(Y^2 - X^n)$, $n \geq 2$, q odd, then $\widehat{\zeta}_R(s)$ is an explicit power series in $q^{-1}, t = q^{-s}$ involving partitions, etc.
- Remark: When $n > 2$, direct matrix count is hard; we proved the above using our formulas for $\zeta_{R^d}(s)$.
- The formulas surprisingly give modular forms and Ramanujan-type series.
- Example: $n = 3$, get $\sum_{n\geq 0} Q^{n^2}/((1-Q)\dots(1-Q^n))\, t^{2n}$, where $Q = q^{-1}$. At $t = \pm 1$, get modular form by Rogers–Ramanujan.

Future goal

An obvious question

Existence of meromorphic continuation of $\widehat{\zeta}_R(s)$ is not known in general, but true in known examples so far.

Further-reaching questions

- How do these richer formulas, modular forms, etc. say about the knot theory associated to the singularities?
- Any hope of exact formulas in the number field case?

Thank you for listening!