In the absence of partitions

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Partitions

- Partitions index many things.
- Representation theory: irreducible representations of S_n.
- Algebraic geometry: Schubert varieties in a Grassmannian.
- Symmetric functions: elements of many bases.

Question

Other interesting models for partitions?

Yes — let's look at the one used to define the Hall polynomial.

Types of abelian p-groups

- By the classification theorem, any finite abelian p-group M is uniquely of the form

$$M = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\lambda_l}\mathbb{Z}, \quad \lambda_1 \geq \cdots \geq \lambda_l > 0.$$

- We say the partition $\lambda := (\lambda_1, \ldots, \lambda_l)$ is the type of M.
- For abelian p-groups $M \subseteq N$, we define the cotype of M in N as the type of M/N.

Hall polynomials

Definition

- Given any partitions λ, μ, ν.
- Define $g^{\lambda}_{\mu,\nu}(p)$ to be the number of subgroups of a fixed type-λ p-group of type μ and cotype ν, for any prime p.
- (Hall) $g^{\lambda}_{\mu,\nu}(p)$ is a polynomial in p, called the Hall polynomial; $g^{\lambda}_{\mu,\nu}(t) = g^{\lambda}_{\nu,\mu}(t)$; $g^{\lambda}_{\mu,\nu}(0)$ is the Littlewood–Richardson coefficient.

Remarks

- The Hall algebra has a basis indexed by partitions and structure constants given by $g^{\lambda}_{\mu,\nu}(t)$.
- The Hall–Littlewood (symmetric) function interpolates many famous symmetric functions. The structure constants are essentially $g^{\lambda}_{\mu,\nu}(t^{-1})$.

Automorphisms

Definition

- Let $a_\lambda(p)$ be the number of automorphisms of an abelian p-group of type λ.
- Important formula:

$$a_\lambda(p) = p^{\sum_{i\geq 1} \lambda_i'^2} \prod_{i\geq 0} (p^{-1}; p^{-1})_{\lambda_i' - \lambda_{i+1}'},$$

  where λ_i' is the i-th column of λ, and $(q;q)_n := (1-q)(1-q^2)\ldots(1-q^n)$.
- In particular, $a_\lambda(p)$ is a polynomial in p.
- $a_\lambda(t)$ plays a role in Hall–Littlewood functions.

Takeaway

Some functions of partitions have algebraic interpretations like this.

More explicit formulas

- $g^{\lambda}_{\mu,\nu}(t)$ has an explicit (though very complicated) formula, and the form involves q-hypergeometric functions.
- Easier special case: if $\lambda = (m^d)$ (a box), $\nu = (m^d) - \mu$, then

$$g^{\lambda}_{\mu,\nu}(t) = \frac{t^{d|\mu|}}{a_\mu(t)} \frac{(t^{-1};t^{-1})_d}{(t^{-1};t^{-1})_{d-\mu'_1}}.$$

- Summations are not too bad:

$$\sum_{\nu} g^{\lambda}_{\mu,\nu}(t) = t^{\sum_{i\geq 1} \mu'_i(\lambda'_i - \mu'_i)} \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_{t^{-1}},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q := (q;q)_n/((q;q)_k(q;q)_{n-k})$. (Warnaar '13)

Summation identities

Interpreting partitions as types can lead to summation identities.

A toy example

- $\sum_{\mu,\nu} g_{\mu,\nu}^{(m^d)}(t) \cdot t^{d|\mu|} \cdot \frac{(t^{-1};t^{-1})_d}{(t^{-1};t^{-1})_{d-\mu'_1}} = t^{md^2}$.
- Proof. Suffices to prove the case $t = p$ is a prime.
- RHS counts homomorphisms $f : (\mathbb{Z}/p^m)^d \to (\mathbb{Z}/p^m)^d$. (They can be given by $d \times d$ matrices over $\mathbb{Z}/p^m$.)
- LHS counts it in a different way.
- Let $M = \mathrm{im}(f)$ and μ be the type of M.
- There are $\sum_{\nu} g_{\mu,\nu}^{(m^d)}(p)$ choices of M.
- If M is fixed, then f is determined by a surjection $(\mathbb{Z}/p^m)^d \to M$.
- By Nakayama's lemma, there are $p^{d|\mu|} \cdot \frac{(p^{-1};p^{-1})_d}{(p^{-1};p^{-1})_{d-\mu'_1}}$ many.

In the absence of partitions

A recipe to generalize

- Recall: partition $\rightsquigarrow$ finite abelian p-group $=$ finite $\mathbb{Z}_p$-module.
- Same story if $\mathbb{Z}_p$ is replaced by any DVR.
- Replace $\mathbb{Z}_p$ by a non-DVR $R \rightsquigarrow$ Replace partitions by finite R-modules.

Question

Does this generalization lead to interesting identities?

Answer

- Sometimes we get identities with summations over R-modules even when they are impossible to index explicitly. (w/ Cheong)
- In special cases, we can, which lead to partition identities, though more convoluted. (w/ Jiang)

Random partitions from random matrices

- The cokernel of $\mathbb{Z}_p$-matrix gives a $\mathbb{Z}_p$-module.
- Thus, a matrix gives a partition by taking the type of the cokernel.
- When the matrix is random, we get a random partition.
- There are many random matrix models: uniformly random matrix, uniformly random symmetric matrix, random $0,1$-matrix (Wood), products of random matrices (van Peski), polynomials of random matrices (Cheong, H.), etc.
- They each produce a random partition with interesting distribution.
- Some have graph-theoretic motivation: symmetric $0,1$-matrix $\rightsquigarrow$ random graph, cokernel $\rightsquigarrow$ sandpile group.

The fact that "probabilities sum up to 1" often produces interesting identities. Direct proof was sometimes found after the probability distribution, often using tools from Hall–Littlewood functions.

Work with Cheong

Overview

- Fix a monic polynomial $P(t)$ in $\mathbb{Z}_p[t]$.
- Let $X \in \mathrm{Mat}_n(\mathbb{Z}_p)$ be uniformly random.
- Question. How does $\mathrm{cok}(P(X))$ (as an abelian p-group) distribute?
- Conjecture (Cheong, H. '21). Proposed a distribution, in which the formula is sensitive to how $P(t)$ is factorized mod p.
- It turns out that one has to remember an additional structure on $\mathrm{cok}(P(X))$! (Cheong, Lee, Kaplan, etc.)

Non-DVR comes into play

- Let $R = \mathbb{Z}_p[t]/P(t)$. Then there is an R-module structure on $\mathrm{cok}(P(X))$.
- t acts on $\mathrm{cok}(P(X))$ by sending $v \bmod \mathrm{im}(P(X))$ to $Xv \bmod \mathrm{im}(P(X))$.

Work with Cheong

Theorem (Cheong, Yu '23)

For any finite R-module M, the probability that $\operatorname{cok}(P(X)) \cong_R M$, as $n \to \infty$, approaches $1/|\operatorname{Aut}_R M| \cdot \prod_{j=1}^{l}(p^{-d_j}; p^{-d_j})_\infty$ if M satisfies a "$b_0 = b_1$" condition, and zero otherwise, where

- *$l, d_1, \ldots, d_l$ are read from the factorization data of $P(t)$ mod p.*
- *"$b_0 = b_1$" condition comes from minimal resolutions and Betti numbers of localizations of M.*

Consequence

The sum of $1/|\operatorname{Aut}_R M|$ over finite R-modules satisfying $b_0 = b_1$ condition is $\prod_{j=1}^{l}(p^{-d_j}; p^{-d_j})_\infty^{-1}$. A non-partition-sum result!

Work with Cheong

Theorem (Cheong, H.)

A similar but different formula holds for an analogous model, in which the random matrix X has a fixed residue class mod p. Moreover, our formula is exact for each n (before taking limit).

The proof relies on understanding a more straightforward model that produces an R-module. Namely, the cokernel of an R-matrix.

Work with Cheong

Theorem (Cheong, H.)

Let R be any complete Noetherian local ring with residue field $\mathbb{F}_q$. Let M be a finite R-module; we have well-defined integers $b_0(M)$, $b_1(M)$ called the Betti numbers of M. Let $n, u \geq 0$ and let X be a uniformly random $n \times (n+u)$ matrix over R. Then the probability that $\operatorname{cok}_R(X) \cong_R M$ is $1/|\operatorname{Aut}_R M| \cdot \prod_{i=u+b_0-b_1+1}^{n+u}(1-q^{-i}) \prod_{i=n-b_0+1}^{n}(1-q^{-i})$ if $n \geq b_0 \geq b_1 - u$, and zero otherwise.

Setting total probability $= 1$ and some elementary work, one can obtain a non-partition-sum analog of Euler's identity:

Corollary

When summed over all finite R-modules M, we have $\sum_M \frac{t^{\ell(M)}}{|\operatorname{Aut}_R M|}(tq^{-1};q^{-1})^{-1}_{b_0(M)-b_1(M)} = (tq^{-1};q^{-1})^{-1}_{\infty}$, where $\ell(M)$ is defined by $q^{\ell}M = |M|$.

Break

Lattice zeta function

Work of Solomon '77

- Consider $L = \mathbb{Z}^d$, visualized as a full lattice in $\mathbb{Q}^d$ (or $\mathbb{R}^d$).
- A sublattice $M \subseteq L$ is a $\mathbb{Z}$-submodule of L of finite index. Write the index as $(L : M)$.
- Question. How many sublattices of given index are there?
- To study this (and its asymptotic), Solomon defined a generating function $\zeta_L(s) = \sum_M (L : M)^{-s}$.
- He found that $\zeta_L(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-d+1)$, where $\zeta(s)$ is the Riemann zeta function.

Relation to partitions

For each prime p, the p-part of $\mathbb{Z}^d/M$ is a finite abelian p-group, which has a type. One can express $\zeta_L(s)$ in terms of partition sums by grouping together all M's that have the same p-type.

Work with Jiang

An analogous setting

- Let $k = \mathbb{F}_q$ be a finite field and $R = R_{2,n} = k[[X,Y]]/(Y^2 - X^n)$, $n \geq 2$. (If both $n = 2m, q$ are even, replace by $Y(Y - X^m)$.)
- What is $Z_{R^d}(t) := \sum_{M \subseteq R^d} t^{[R^d:M]}$, summed over R-submodules M with $[R^d : M] := \dim_k R^d/M < \infty$?

Previous work

- R is not a DVR, so M can no longer be classified by partitions.
- Nevertheless, such functions (for R in more generality) are known to have nice general properties. (Bushnell, Reiner '80s)
- When $d = 1$, explicit formulas are expected to have knot-theoretic interpretation. (Oblomkov, Rasmussen, Shende '18)
- When $d = 1$ and $R = k[[X,Y]]/(Y^m - X^n)$ with m, n coprime, we get generalized q,t-Catalan. (Gorsky, Mazin '13)

Our formulas

Theorem (H., Jiang)

For $R = R_{2,2m+1}$, $m \geq 1$, *then* $(t;q)_d Z_{R^d}(t)$ *is the* q,t*-polynomial* $C_{m,d} :=$

$$\sum_{\mu \subseteq (m^d)} g^{(m^d)}_{\mu,(m^d)-\mu}(q)\,(q^d t^2)^{|\mu|}.$$

For $R = R_{2,2m}$, $m \geq 1$, *then* $(t;q)_d^2 Z_{R^d}(t)$ *is the* q,t*-polynomial* $N_{m,d} :=$

$$\sum_{\lambda,\mu,\nu \subseteq (m^d)} g^{(m^d)}_{\lambda,(m^d)-\lambda}(q)\, g^{\lambda}_{\mu,\nu}(q)\, t^{|\lambda|} (q^d t)^{|\lambda|-|\mu|} (t;q)^2_{d-\lambda'_m} \frac{(q^{-1};q^{-1})_{\lambda'_m}}{(q^{-1};q^{-1})_{\mu'_m}}.$$

Remark

One can make both formulas explicit by rewriting $g^{(m^d)}_{\lambda,(m^d)-\lambda}$ and $\sum_\nu g^{\lambda}_{\mu,\nu}$.

Tables

Case of $Y^2 = X^3$:

d	$C_{1,d}(t,q)$
0	1
1	$1 + qt^2$
2	$1 + (q^3 + q^2)t^2 + q^4t^4$
3	$1 + (q^5 + q^4 + q^3)t^2 + (q^8 + q^7 + q^6)t^4 + q^9t^6$
4	$1 + (q^7 + q^6 + q^5 + q^4)t^2 + (q^{12} + q^{11} + 2q^{10} + q^9 + q^8)t^4 + (q^{15} + q^{14} + q^{13} + q^{12})t^6 + q^{16}t^8$

Table: $C_{m,d}(t,q)$ with $m = 1$

Tables

Case of $Y^2 = X^2$:

d	$N_{1,d}(-t, q)$
0	1
1	$1 + t + qt^2$
2	$1 + (q+1)t + (q^3 + q^2 + q)t^2 + (q^3 + q^2)t^3 + q^4t^4$
3	$1 + (q^2 + q + 1)t + (q^5 + q^4 + 2q^3 + q^2 + q)t^2 + (q^6 + 2q^5 + 2q^4 + 2q^3)t^3 + (q^8 + q^7 + 2q^6 + q^5 + q^4)t^4 + (q^8 + q^7 + q^6)t^5 + q^9t^6$
4	$1 + (q^3 + q^2 + q + 1)t + (q^7 + q^6 + 2q^5 + 2q^4 + 2q^3 + q^2 + q)t^2 + (q^9 + 2q^8 + 3q^7 + 4q^6 + 3q^5 + 2q^4 + q^3)t^3 + (q^{12} + q^{11} + 3q^{10} + 3q^9 + 4q^8 + 3q^7 + 3q^6 + q^5)t^4 + (q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 3q^9 + 2q^8 + q^7)t^5 + (q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + q^{10} + q^9)t^6 + (q^{15} + q^{14} + q^{13} + q^{12})t^7 + q^{16}t^8$

Table: $N_{m,d}(-t, q)$ with $m = 1$

Combinatorial properties

Functional equation

A general theorem we prove implies that if $F(t,q) = C_{m,d}$ or $N_{m,d}$, then

$$F(q^{-d}t^{-1}, q) = (q^d t^2)^{-dm} F(t,q).$$

Open problem

Give a direct proof of the above for $N_{m,d}$. Open for $m \geq 2$.

Positivity

$C_{m,d}(\pm t, q) \in \mathbb{N}[t,q]$ is clear from the formula. We expect that $N_{m,d}(-t,q) \in \mathbb{N}[t,q]$ and there is a nontrivial proof when $m = 1$.

Open problem

Prove or disprove: $N_{m,d}(-t,q) \in \mathbb{N}[t,q]$ for $m \geq 2$. How about unimodality?

Thank you for listening!

One more table

Case of $Y^2 = X^6$ (you can figure out the whole $d = 3$ by functional equ):

d	$N_{3,d}(-t, q)$
0	1
1	$1 + t + qt^2 + qt^3 + q^2t^4 + q^2t^5 + q^3t^6$
2	$1 + (q+1)t + (q^3+q^2+q)t^2 + (q^4+2q^3+q^2)t^3 + (q^6+q^5+2q^4+q^3)t^4 + (q^7+2q^6+2q^5+q^4)t^5 + (q^9+q^8+2q^7+2q^6+q^5)t^6 + (q^9+2q^8+2q^7+q^6)t^7 + (q^{10}+q^9+2q^8+q^7)t^8 + (q^{10}+2q^9+q^8)t^9 + (q^{11}+q^{10}+q^9)t^{10} + (q^{11}+q^{10})t^{11} + q^{12}t^{12}$
3	$1 + (q^2+q+1)t + (q^5+q^4+2q^3+q^2+q)t^2 + (q^7+2q^6+3q^5+2q^4+2q^3)t^3 + (q^{10}+q^9+3q^8+3q^7+4q^6+2q^5+q^4)t^4 + (q^{12}+2q^{11}+4q^{10}+4q^9+5q^8+3q^7+2q^6)t^5 + (q^{15}+q^{14}+3q^{13}+4q^{12}+6q^{11}+5q^{10}+5q^9+2q^8+q^7)t^6 + (q^{16}+3q^{15}+5q^{14}+7q^{13}+6q^{12}+6q^{11}+3q^{10}+2q^9)t^7 + (q^{18}+2q^{17}+5q^{16}+6q^{15}+8q^{14}+6q^{13}+5q^{12}+2q^{11}+q^{10})t^8 + (q^{19}+4q^{18}+6q^{17}+8q^{16}+7q^{15}+6q^{14}+3q^{13}+2q^{12})t^9 + (q^{21}+2q^{20}+5q^{19}+6q^{18}+8q^{17}+6q^{16}+5q^{15}+2q^{14}+q^{13})t^{10} + (q^{22}+3q^{21}+5q^{20}+7q^{19}+6q^{18}+6q^{17}+3q^{16}+2q^{15})t^{11} + (q^{24}+q^{23}+3q^{22}+4q^{21}+6q^{20}+5q^{19}+5q^{18}+2q^{17}+q^{16})t^{12} + (q^{24}+$