A generalization of a zeta function of Cohen–Lenstra

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Motivation 1: Cohen–Lenstra heuristics

Let R be a Dedekind domain with only finite quotient fields. (For example, $R = \mathbb{Z}$.) Cohen and Lenstra considered the following function

$$\widehat{\zeta}_R(s) = \sum_M \frac{1}{|\operatorname{Aut} M|} |M|^{-s}, \tag{1}$$

where M runs over all isomorphism classes of finite-cardinality R-modules.

They proved formulas and theorems for $\widehat{\zeta}_R(s)$, which are crucial in their important work on the distribution of class groups of quadratic fields, where they proposed the **Cohen–Lenstra heuristics**.

Motivation 2: Rogers–Ramanujan

Consider the power series in x and q:

$$F(x; q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} x^n. \qquad (2)$$

The two **Rogers–Ramanujan identities** say that $F(1; q)$ and $F(q; q)$ each equals to an infinite product.

We will explain why $F(x; q)$ can be considered the simplest instance of a generalization of $\widehat{\zeta}_R(s)$.

Motivation 3: Matrix enumeration

We will connect $\widehat{\zeta}_R(s)$ to the following combinatorial results.

- Fine and Herstein (1958) counted the number of nilpotent $n \times n$ matrices over $\mathbb{F}_q$.
- Fine and Feit (1960) counted pairs of commuting $n \times n$ matrices over $\mathbb{F}_q$.
- Fulman and Guralnick (2018) counted pairs of commuting $n \times n$ nilpotent matrices over $\mathbb{F}_q$.

More importantly, each of these counts forms a nice generating function.

We will give a common generalization of $\widehat{\zeta}_R(s)$ and these generating functions.

Motivation 4: Moduli spaces

We will also connect $\widehat{\zeta}_R(s)$ to the following recent results:

- [Behrend and Bryan] and [Bryan and Morrison] proved formulas for a generating function for the *motivic class in the Grothendieck ring* of the moduli stack of finite-length coherent sheaves over $\mathbb{C}^2$.
- Moschetti and Ricolfi classified small-length coherent sheaves over $\mathbb{C}^2$ and found their automorphism groups, refining the work above.

A unifying framework

Let R be a commutative ring with only finite quotient fields. We define the **Cohen–Lenstra zeta function** of R to be

$$\widehat{\zeta}_R(s) = \sum_M \frac{1}{|\operatorname{Aut} M|} |M|^{-s}, \tag{3}$$

where M runs over all isomorphism classes of finite-cardinality R-modules. When R is Dedekind, this becomes Cohen–Lenstra's construction.

More generally, let X be a scheme whose residue fields at closed points are all finite. We define the **Cohen–Lenstra zeta function** of X to be

$$\widehat{\zeta}_X(s) = \sum_M \frac{1}{|\operatorname{Aut} M|} |H^0(X; M)|^{-s}, \tag{4}$$

where M runs over all isomorphism classes of finite-length coherent sheaves on X. (Note that such sheaves are torsion and supported at finitely many closed points of X.)

Existing work in the new context

The results mentioned before are statements about different instances of the Cohen–Lenstra zeta function.

- The construction $\widehat{\zeta}_X(s)$ for a scheme X is a direct generalization of Cohen–Lenstra's $\widehat{\zeta}_R(s)$ because $\widehat{\zeta}_R(s) = \widehat{\zeta}_{\mathrm{Spec}\,R}(s)$.
- The series $F(x; q)$ in the Rogers–Ramanujan identities is essentially the Cohen–Lenstra zeta function of a single point. (So even $\widehat{\zeta}_{\mathbb{F}_q}(s)$ is not boring!)
- The three matrix-enumeration results are equivalent to giving formulas to $\widehat{\zeta}_R(s)$ for $R = \mathbb{F}_q[[t]], \mathbb{F}_q[u, v], \mathbb{F}_q[[u, v]]$ respectively.
- The work about the motivic class of a moduli stack is essentially computing $\widehat{\zeta}_X(s)$ where $X = \mathbb{A}^2$ is the plane.

Goal questions

The existing work and our work (to be mentioned) suggest that the Cohen–Lenstra zeta function $\widehat{\zeta}_R(s)$ (or $\widehat{\zeta}_X(s)$) is interesting in its own right. We will explore the following questions:

1. How does it behave algebraically and analytically in general?
2. What do the properties of $\widehat{\zeta}_X(s)$ reveal about the geometry of X?

- (Euler product) The product being over all closed points of X, we have

$$\widehat{\zeta}_X(s) = \prod_p \widehat{\zeta}_{\mathcal{O}_{X,p}}(s) = \prod_p \widehat{\zeta}_{\widehat{\mathcal{O}}_{X,p}}(s) \tag{5}$$

- (Known cases) Formulas for $\widehat{\zeta}_X(s)$ are implicit in the motivic work of [Behrend and Bryan] and [Bryan and Morrison] if X is a smooth curve or a smooth surface over $\mathbb{F}_q$. Moreover, in these cases, all factors in the product above are known.

We will address singular cases (which require different methods), keeping in mind that the question is local due to the Euler product.

Implications of the properties

If X is a singular curve or surface over $\mathbb{F}_q$, the Euler product of $\widehat{\zeta}_X(s)$ splits into known factors ($\widehat{\zeta}_{\mathcal{O}_{X,p}}(s)$ for smooth points p) and mysterious factors ($\widehat{\zeta}_{\mathcal{O}_{X,p}}(s)$ for singular points p).

In other words, $\widehat{\zeta}_X(s)$ is really about the singularities of X. We can partially answer a goal question:

Q What does the property of $\widehat{\zeta}_X(s)$ reveal about the geometry of X?

A Its factor, namely, $\widehat{\zeta}_{\mathcal{O}_{X,p}}(s)$, is an **invariant** attached to the singularity at p. It is a datum that can be added to the classification of singularities, though it is not yet clear what it means geometrically.

Due to the importance of this factor, denote $\widehat{\zeta}_{X,p}(s) := \widehat{\zeta}_{\mathcal{O}_{X,p}}(s)$.

Main conjecture

We propose that the poles of $\widehat{\zeta}_{X,p}(s)$ encode geometry about the resolution of singularity at p. To state the conjecture, we consider the following power series in x for a scheme X over $\mathbb{F}_q$:

$$\widehat{Z}_{X/\mathbb{F}_q}(x) = \sum_M \frac{1}{|\operatorname{Aut} M|} x^{\dim_{\mathbb{F}_q} H^0(X;M)} \tag{6}$$

We have $\widehat{\zeta}_X(s) = \widehat{Z}_{X/\mathbb{F}_q}(q^{-s})$. We will drop $\mathbb{F}_q$ in the notation if the ground field is implicit in the context.

Conjecture (H.)

Assume that p is a singular $\mathbb{F}_q$-point on a reduced $\mathbb{F}_q$-curve X that admits a resolution singularity $\pi : \widetilde{X} \to X$. Then $\widehat{Z}_{X,p}(x)$ has a meromorphic continuation to $\mathbb{C}$ given by

$$\widehat{Z}_{X,p}(x) = \left((1 - q^{-1}x)(1 - q^{-2}x)\ldots\right)^{-|\pi^{-1}(p)|} H_{X,p}(x) \tag{7}$$

for some entire function $H_{X,p}(x)$.

- The power series $H_{X,p}(x)$ only depends on the singularity (X, p). Thus $H_{X,p}(x)$ is another **invariant** of the singularity (X, p).
- The conjecture claims that $H_{X,p}(x)$ is always entire.
- The function $\widehat{Z}_{X,p}(x)$ is already an Euler factor, which does not promise any further factorization a priori. The unusual claim that it can be further factorized is not a consequence of the Euler product.
- The conjecture implies that $\widehat{Z}_{X,p}(x)$ has a meromorphic continuation to all of $\mathbb{C}$, which is nontrivial.
- The conjecture gives a complete description of the poles of $\widehat{Z}_{X,p}(x)$ in terms of the **branching number** of p.

The main conjecture is equivalent to an elegant global statement.

Conjecture (H.)

Let X be a reduced curve over $\mathbb{F}_q$, and let $\widetilde{X}$ be a resolution of singularity of X. Then $\widehat{Z}_X(x)$ has a meromorphic continuation to $\mathbb{C}$ given by

$$\widehat{Z}_X(x) = \widehat{Z}_{\widetilde{X}}(x) H_X(x) \tag{8}$$

for some entire function $H_X(x)$.

- (Main result) We prove that the conjecture holds for a type of singularity, namely, the **(ordinary) node**. This is the first and only proven case of this conjecture so far.

- As a heuristical evidence, an analogous statement holds for an analogous generating function of *Hilbert schemes* if the singularity is **planar** (having a model on a plane curve), or more generally, **Gorenstein**. (work of Göttsche and Shende)

Main result

Theorem (H.)

Let $Y_q = \operatorname{Spec} \mathbb{F}_q[u, v]/(uv)$ be the union of two intersecting lines. Then the conjecture holds for Y_q, namely, $\widehat{Z}_{Y_q}(x)$ is of the form

$$\widehat{Z}_{Y_q}(x) = \left((1 - x)(1 - q^{-1}x)(1 - q^{-2}x)\ldots\right)^{-2} H_q(x) \qquad (9)$$

where $H_q(x)$ is entire.

In particular, this result shows that the conjecture holds for curves with only (ordinary) nodal singularities.

Main result: combinatorics

We prove the main result by proving the following combinatorial result, which is interesting its own right.

Theorem (H.)

We have an identity of formal power series in x:

$$\sum_{n=0}^{\infty} \frac{|\{(A, B) : A, B \in \mathrm{Mat}_n(\mathbb{F}_q), AB = BA = 0\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n \tag{10}$$
$$= \left((1-x)(1-q^{-1}x)(1-q^{-2}x)\ldots\right)^{-2} H_q(x)$$

where

$$H_q(x) := \sum_{k=0}^{\infty} \frac{q^{-k^2} x^{2k}}{(1-q^{-1})\ldots(1-q^{-k})}(1-q^{-k-1}x)(1-q^{-k-2}x)\ldots \tag{11}$$

We remark that the hard part of the proof is to find this particular expression of $H_q(x)$.

Remarks

- It is unknown if the main result has a geometric proof. The only known proof that $H_q(x)$ is entire is its explicit combinatorial formula.
- The evidence so far suggests that there are mysteries about the stack of finite-length coherent sheaves that we don't understand. We hope that an attempt to prove the conjecture can yield deeper understandings about its geometry.
- The conjecture can always be "tested" by converting it into a matrix counting question. However, this question is still too hard even for a computer test.
- The conjecture, if true, would imply the following asymptotics:

$$|\{(A, B) : A, B \in \mathrm{Mat}_n(\mathbb{F}_q), A^2 = B^d, AB = BA\}|$$
$$\sim \begin{cases} C(d,q)\, q^{n^2}, & d \text{ odd;} \\ C(d,q)\, nq^{n^2}, & d \text{ even,} \end{cases}$$

for some $C(d,q) > 0$ as $n \to \infty$, assuming q is odd.

Break

We continue exploring the two goal questions:

1. How does $\widehat{Z}_X(x)$ behave algebraically and analytically in general?
2. What do the properties of $\widehat{Z}_X(x)$ reveal about the geometry of X?

Now, in light of the conjecture, for a curve singularity (X, p), we can ask a refined version of the second question:

3. What do the properties of the (conjecturally) holomorphic factor $H_{X,p}(x)$ reveal about the geometry at p?

General analytic behavior

1. How does $\widehat{Z}_X(x)$ behave algebraically and analytically in general?

Partial answer: It appears that the analytic behavior of $\widehat{Z}_X(x)$ is determined by $\dim X$ more than anything else (such as smoothness and reducedness).

Let X be an $\mathbb{F}_q$-variety that is not necessarily reduced, then

- If $\dim X = 0$, then $\widehat{Z}_X(x)$ appears to be entire.
- If $\dim X = 1$, then $\widehat{Z}_X(x)$ appears to have a meromorphic continuation to all of $\mathbb{C}$, whose poles carry important geometry information (e.g. the main conjecture).
- If $\dim X = 2$, then $\widehat{Z}_X(x)$ appears to be meromorphic on the unit disk (and cannot be extended further).
- If $\dim X \geq 3$, then $\widehat{Z}_X(x)$ appears to be divergent for $x \neq 0$.

On the holomorphic factor

③ For a curve singularity (X, p), what do the properties of the holomorphic factor $H_{X,p}(x)$ (in the main conjecture) reveal about the geometry at p?

It appears that

The function $H_{X,p}(x)$ for a general curve singularity (X, p) should share some features with the **partial theta function**.

The partial theta function $\Theta_p(x; q^{-1}) := \sum_{n \geq 0} q^{-n^2} x^n$ has the following features:

- It has a functional equation.

- It has *smooth coefficients*, namely, $\dfrac{a_n^2}{a_{n-1} a_{n+1}}$ converges as $n \to \infty$.

On the holomorphic factor

Many analytic properties of the partial theta function are consequences of its smooth coefficients, as is shown in the work of Nguyen and Vishnyakova, etc. Having smooth coefficients is thus the key feature to look for in order to compare $H_{X,p}(x)$ with Θ_p.

We conjecture the following based on numerical observations:

Conjecture (H.)

Let $\sum_{n=0}^{\infty} a_n x^n$ be the series $H_q(x)$ in the main result. Then its even-degree terms and odd-degree terms have smooth coefficients, and moreover,

$$\lim_{n \to \infty} \frac{a_{2n}^2}{a_{2n-2} a_{2n+2}} = \lim_{n \to \infty} \frac{a_{2n+1}^2}{a_{2n-1} a_{2n+3}} = q^2 \qquad (12)$$

Final takeaways

- We define the Cohen–Lenstra zeta function, a generalization of a function defined by Cohen and Lenstra.

- The generalization is a common framework that unifies several problems from module classification, q-series, matrix counting, and moduli spaces.

- We explore how properties of the Cohen–Lenstra zeta function of a variety depend on its geometry. We find that they mostly depend on the dimension and the singularities, but there are still mysteries.

- The most notable pattern is that the resolution of a curve singularity appears to determine the pole of the Cohen–Lenstra zeta function. The main result verifies the case of nodal singularity, supporting the guess that the pattern holds in general.

Thank you!