

# Topics on Polynomial Equations in Noncommutative Rings and Motivic Aspects of Moduli Spaces

by

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*To my parents and my siblings*

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# ABSTRACT

We investigate three topics that are motivated by the study of polynomial equations in noncommutative rings. These topics have distinct flavors, ranging from number theory, combinatorics to topology.

As the first topic, we study a noncommutative analogue of a classical theorem in number theory that the unit equation  $x + y = 1$ , where both  $x$  and  $y$  belong to a given finitely generated subgroup of the multiplicative group of nonzero complex numbers, has only finitely many solutions. We show that if  $x$  and  $y$  are nonzero quaternions expressible as certain products, then the unit equation  $x + y = 1$  on the (noncommutative) quaternion algebra only has finitely many solutions. We also give a natural application to the study of iterations of self-maps on abelian varieties whose endomorphism rings lie inside the quaternion algebra.

As the second topic, we count the numbers of solutions of several equations on the ring of  $n$  by  $n$  matrices over a finite field. We investigate the combinatorial behaviors of these counts by giving generating functions. Each of these counts can be viewed as the point count (over a finite field) of a space that parametrizes finite-dimensional modules over a certain algebra that arises from algebraic geometry. The connection between the count and the underlying geometry is also discussed. In Chapter III, we count pairs of mutually annihilating matrices  $AB = BA = 0$  over a finite field; the underlying geometry is a nodal singularity on an algebraic curve. In Chapter IV, we count pairs of matrices satisfying  $AB = \zeta BA$ , where  $\zeta$  is a root of unity in a finite field; the underlying geometry is the quantum plane.

As the third topic, we focus on the configuration space, which is the space that parametrizes unordered tuples of distinct points on a base space. We give several results that state that certain combinatorial behaviors of some geometric invariants (namely, Betti numbers and mixed Hodge numbers) of configuration spaces are analogous to the well-known behavior of the point counts of configuration spaces over finite fields. In Chapter V, we give a rational generating function, which is essentially a zeta function, that encodes Betti and mixed Hodge numbers of configuration spaces of a punctured elliptic curve over  $\mathbb{C}$ . In Chapter VI, we describe the effect of puncturing a point from the base space on the Betti and mixed Hodge numbers of the configuration spaces, under a certain assumption.

# CHAPTER I

## Introduction

### I.1: History and motivation

The study of polynomial equations has been a central subject of mathematics since antiquity. We list two classical problems about polynomial equations of distinct flavors.

**Example I.1.1.** Find all solutions of the Pythagorean equation  $x^2 + y^2 = z^2$  where  $x, y, z$  are positive integers. This problem has been considered since the discovery of the simplest right triangles  $((3, 4, 5)$  and  $(5, 12, 13)$ , etc.) and the Pythagorean theorem. It turns out that there are infinitely many solutions  $(x, y, z)$  that are coprime (namely,  $\gcd(x, y, z) = 1$ ), and it was known to Diophantus of Alexandria in 250 CE that every solution  $(x, y, z)$  of the Pythagorean equation is of the form  $(2mn, m^2 - n^2, m^2 + n^2)$  where  $m, n$  are integers. Motivated by the Pythagorean equation, Pierre de Fermat (c. 1637 CE) considered an analogous equation  $x^n + y^n = z^n$  where  $n \geq 3$  is a given integer. The conjecture that the Fermat equation  $x^n + y^n = z^n$  has no integer solution with  $xyz \neq 0$  for any  $n \geq 3$ , known as Fermat's Last Theorem, was not proven until 1994 by Andrew Wiles.

**Example I.1.2.** A conic section (or simply called a conic) is a curve in the plane  $\mathbb{R}^2$  given as the set of all solutions to a quadratic equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , for some parameters  $A, B, C, D, E, F$ . It was known to the ancient Greeks that conic sections can be classified into three types: ellipses, parabolae and hyperbolae. The geometry of conics has been extensively studied by mathematicians including Archimedes (died c. 212 BCE).

The Pythagorean equation asks for integer solutions. Such equations in integers (or more generally, equations in rational numbers as well) are delicate in that the existence of every solution is miraculous to some extent, since we only allow a discrete selection of solutions. The existence and the finiteness of solutions are often interesting problems to ask about such an equation, and the answer is typically sensitive to any slight change of the equation. The subject that studies polynomial equations in integers or rational numbers is called

*Diophantine equations*, and tackling problems in this subject generally requires methods in number theory and discrete mathematics.

The defining equation for a conic section asks for real solutions; the solution set is precisely the conic. Such equations in real numbers (or more generally, equations in complex numbers as well) tend to have a continuum of solutions. Therefore, rather than the existence of solutions, the structure (especially the geometry and the topology) of the solution set is a more typical object of study. The subject that studies the solution set of polynomial equations in real or complex numbers is called *algebraic geometry*, and such a solution set is called an *algebraic variety*.

Remarkably, though the above two types of equations have distinct flavors, they have many deep connections, which fuel the development of a large part of modern mathematics. The proof of Fermat’s Last Theorem is a good example of such a connection; algebraic geometry played an essential role in the proof of Wiles. We now use Examples [I.1.1](#) and [I.1.2](#) to illustrate a simpler instance of such a connection. We recall that the Pythagorean equation has infinitely many solutions, and furthermore, the solutions can be parametrized by two integers  $m$  and  $n$ . This fact is essentially a geometric fact about conics, which after a rephrasing in the language of algebraic geometry, states that a conic, understood as a nondegenerate quadratic curve inside the complex projective plane  $\mathbb{CP}^2$ , is isomorphic to the projective line  $\mathbb{CP}^1$  as a complex variety. Since the set  $C$  of complex solutions of  $x^2 + y^2 = z^2$  can be in fact viewed as a conic, the isomorphism above provides a map  $\mathbb{CP}^1 \rightarrow C$ , and it is precisely the parametrization described by Diophantus. More precisely, the said parametrization of the quadratic curve  $x^2 + y^2 = z^2$  in  $\mathbb{CP}^2$  is defined by

$$\begin{aligned} \mathbb{CP}^1 &\hookrightarrow \mathbb{CP}^2 \\ [u : v] &\mapsto [2uv : u^2 - v^2 : u^2 + v^2]. \end{aligned} \tag{I.1.1}$$

## I.2: Overview of the thesis

The starting point of this thesis is the study of polynomial equations in *noncommutative* rings. The two flavors of equations we have discussed above both involve equations in integers, rational numbers, real numbers or complex numbers; these rings all have commutative multiplication. The noncommutative rings we are going to consider are the quaternion algebra and the matrix algebra.

Chapter [II](#) considers a Diophantine equation called the “unit equation” on the quaternion algebra. The classically considered unit equation on  $\mathbb{C}$  is an equation of the form  $x + y = 1$ , where  $x$  and  $y$  are nonzero complex numbers (hence the name “unit”) generated from a

given finite set  $S$  of numbers through multiplication and division. A classical theorem says that the set of solutions is always finite. For example, if  $S = \{2, 3, -1\}$ , then  $x$  and  $y$  are only allowed to be rational numbers whose numerators and denominators are powers of 2 or 3. The classical theorem is then equivalent to a more straightforward statement that there are only finitely many pairs of a power of 2 and a power of 3 that differ by 1. In fact,  $(2, 1), (2, 3), (4, 3), (8, 9)$  are all such pairs. One can see that the unit equation is a delicate equation that reveals deep information about the interaction between the basic operations of numbers, addition and multiplication. In fact, the unit equation on  $\mathbb{C}$  is an important Diophantine equation, and it has had wide applications in Diophantine geometry, arithmetic dynamics and related areas.

The quaternion algebra  $\mathbb{H}$  consists of elements  $a + bi + cj + dk$  with real numbers  $a, b, c, d$  and formal symbols  $i, j, k$ . The multiplication on  $\mathbb{H}$  is determined by the rules  $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ . In particular, the multiplication on  $\mathbb{H}$  is noncommutative. The main result of Chapter II states that the equation  $x + y = 1$  on  $\mathbb{H}$  has only finitely many solutions such that  $x$  and  $y$  are nonzero elements of  $\mathbb{H}$  expressible as a (noncommutative) product following a certain pattern and using elements from a given finite set. This result is the first such finiteness result in a noncommutative setting. The relaxing of the commutativity assumption introduces essential difficulties because many important tools (such as  $p$ -adic valuations) that are crucial in the study of Diophantine equations are no longer available in a noncommutative setting.

Chapters III and IV, on the other hand, consider solution sets of certain polynomial equations in the matrix ring  $\text{Mat}_n(\mathbb{C})$  of  $n$  by  $n$  complex matrices. The solution sets in question are algebraic varieties over  $\mathbb{C}$ , and the focus of our study of their geometry is how they are composed of simpler varieties (such as affine planes  $\mathbb{C}^n$ ) via taking disjoint unions and complements. In a precise terminology, we are computing their *motivic classes* in the *Grothendieck ring* of varieties. To ease the statement, we choose to state our results in terms of point counting over a finite field; for the precise statements, we refer the reader to Sections I.3.2 and I.3.3. The nature of the problems turns out to be combinatorial, and nothing essential is lost after the restatement. Both results can be viewed as generalizations of a classical result of Feit and Fine [FF60] about enumerating pairs of commuting matrices: let  $\text{Mat}_n(\mathbb{F}_q)$  denote the set of  $n$  by  $n$  matrices over the finite field  $\mathbb{F}_q$  of  $q$  elements, then the counting of pairs of commuting matrices is determined by the following generating function

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \text{Mat}_n(\mathbb{F}_q) : AB = BA\}|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}}, \quad (\text{I.2.1})$$

where  $|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$ .

The algebraic varieties studied in Chapters III and IV are a special type of variety, called a *moduli space*. A moduli space is a space that parametrizes all possible structures of a certain kind. For instance, the projective space  $\mathbb{P}^1$  is the moduli space of straight lines that pass through the origin. The algebraic variety

$$\{A, B \in \text{Mat}_n(\mathbb{C}) : AB = BA\} \tag{I.2.2}$$

considered by Feit and Fine is in fact the moduli space of  $n$ -dimensional modules over the polynomial ring  $\mathbb{C}[u, v]$ . The variety

$$\{A, B \in \text{Mat}_n(\mathbb{C}) : AB = BA = 0\} \tag{I.2.3}$$

considered in Chapter III is the moduli space of  $n$ -dimensional modules over the quotient ring  $\mathbb{C}[u, v]/(uv)$ , and the variety

$$\{A, B \in \text{Mat}_n(\mathbb{C}) : AB = \zeta BA\} \tag{I.2.4}$$

(where  $\zeta$  is a root of unity) considered in Chapter IV is the moduli space of  $n$ -dimensional modules over the *quantum plane*, a noncommutative algebra that can be viewed as a quantum deformation of the polynomial ring  $\mathbb{C}[u, v]$ .

The results in Chapters III and IV reflect a theme that the algebraic variety defined by a polynomial equation on the matrix algebra tends to be a moduli space, and the point counting (and related motivic aspects) of a moduli space tends to have good combinatorial properties.

Chapters V and VI focus on another important moduli space, namely, the configuration space of points on an algebraic variety over  $\mathbb{C}$ . The configuration space  $\text{Conf}^n(X)$  of a variety  $X$  parametrizes all unordered tuples of  $n$  distinct points of  $X$ . It turns out that the motivic class (and thus the point counting over finite fields) of  $\text{Conf}^n(X)$  is well understood, see [VW15]. The focus of Chapters V and VI lies in other geometric invariants, namely, Betti numbers and mixed Hodge numbers, of the configuration spaces. These invariants are not *motivic*, in the sense that they cannot be determined by the motivic class in the Grothendieck ring of varieties. Informally speaking, a nonmotivic invariant of a variety  $X$  not only depends on what simpler spaces make up  $X$ , but also depends on how the simpler spaces glue together to form  $X$ . For example, the projective line  $\mathbb{C}\mathbb{P}^1$  equals  $\mathbb{C} \cup \{\text{pt}\}$  as a disjoint union, but the point  $\{\text{pt}\}$  is not “separated” from the copy of  $\mathbb{C}$  as in the case of the topological disjoint union  $\mathbb{C} \sqcup \{\text{pt}\}$ . The 0-th Betti number  $h^0$  can tell the difference: we have  $h^0(\mathbb{C}\mathbb{P}^1) = 1$  but  $h^0(\mathbb{C} \sqcup \{\text{pt}\}) = 2$ .

However, these nonmotivic invariants of configuration spaces sometimes have good combinatorial properties similar to the known ones possessed by motivic invariants. A theorem of Arnol'd [Arn69] shows that the Betti numbers of the configuration spaces  $\text{Conf}^n(\mathbb{C})$  of a plane follow the “same” pattern as the point counts of configuration spaces  $\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)$  of the affine line over a finite field; the point count of  $\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)$  is actually the number of monic square-free polynomials of degree  $n$  over  $\mathbb{F}_q$ , which is known to be  $q^n - q^{n-1}$  when  $n \geq 2$  and  $q^n$  when  $n = 0, 1$ . In Chapter V, we prove that the Betti and mixed Hodge numbers of the configuration spaces of a punctured elliptic curve over  $\mathbb{C}$  can be read off of a rational generating function similar to the one encoding the point counts of the configuration spaces of a punctured elliptic curve over a finite field. The precise statement involves a special degree shifting  $w(i)$  (specified in that chapter), which reveals important geometry about these configuration spaces. The key idea that recovers a nonmotivic invariant from a motivic one is *purity*, a favorable condition about the mixed Hodge numbers that happens to hold for the configuration spaces of the complex plane or a punctured elliptic curve over  $\mathbb{C}$ . In Chapter VI, we show that puncturing a point from the base space has an effect on the Betti and mixed Hodge numbers of the configuration spaces that is analogous to its effect on the motivic classes of configuration spaces, under a certain assumption on the base space. One notable assumption is that the base space must be noncompact. Our result generalizes and refines a result of Kallel [Kal08]. In the setting of Chapter VI, the purity condition no longer holds, and the nonmotivic main result requires a completely separate proof from its motivic analogue.

### I.2.1: Organization of the thesis

In the following section, we give an individual overview of each of the chapters. After that, we put the verbatim copies of the papers [Hua20b] (Chapter II), [Hua21] (Chapter III), [Hua22] (Chapter IV), [CH22] (Chapter V, joint with G. Cheong) and [Hua20a] (Chapter VI), each occupying one chapter.

## I.3: Overview of each chapter

We give an overview of each chapter that is self-contained except that the overviews of Chapter V and Chapter VI require a general discussion in Section I.3.4

### I.3.1: Chapter II: Unit equations on quaternions

Consider the quaternion algebra  $\mathbb{H}$  over  $\mathbb{R}$  defined as  $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$  with the standard multiplication law  $i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$  (note that

multiplication on  $\mathbb{H}$  is not commutative). A unit equation on  $\mathbb{H}$  is an equation of the form  $x + y = 1$ , where  $x, y$  are elements taken from a certain subset of the multiplicative group  $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$ . A typical question to ask about the unit equation is whether the number of solutions is finite. Our main theorem states that the finiteness statement holds under a certain constraint for  $x$  and  $y$ . In the following statement, the algebra  $\mathbb{H}_a$  of algebraic quaternions consists of elements  $a + bi + cj + dk$  where  $a, b, c, d$  are real algebraic numbers, a semigroup is a subset that contains 1 and is closed under multiplication, and the norm of a quaternion  $a + bi + cj + dk$  is defined as  $a^2 + b^2 + c^2 + d^2$ .

**Theorem I.3.1** (Theorem II.1.1). *Let  $\Gamma_1, \Gamma_2$  be semigroups of  $\mathbb{H}_a^\times$  generated by finitely many elements of norms greater than 1, and fix  $a, a', b, b' \in \mathbb{H}_a^\times$ . If  $\Gamma_1$  is commutative, then the equation*

$$af a' + bg b' = 1 \tag{I.3.1}$$

*has only finitely many solutions with  $f \in \Gamma_1$  and  $g \in \Gamma_2$ . In fact, for solutions  $(f, g)$ , we have effectively computable upper bounds for  $|f|, |g|$  that depend only on  $a, a', b, b'$  and generators of  $\Gamma_1, \Gamma_2$ .*

A classical result states that the equation  $x + y = 1$  has only finitely many solutions in a given finitely generated semigroup of the multiplicative group  $\mathbb{C}^\times$  of complex numbers. Theorem I.3.1 can thus be viewed as an analogue of the classical result in the noncommutative algebra  $\mathbb{H}$ .

We emphasize that in Theorem I.3.1, even though  $\Gamma_1$  is commutative, the semigroup  $\Gamma_2$  need not be, and none of the elements  $a, f, a', b, g, b'$  need to commute with each other. The novelty of our result lies in that we remove (the majority of) the commutativity assumption, which plays an essential role in the proof of the classical result about the unit equation on  $\mathbb{C}$ . We manage to reduce the problem to a finiteness problem about an equation on  $\Gamma_1$  only (see Theorem II.1.3), after which the classical approaches apply since  $\Gamma_1$  is commutative.

The study of equations on noncommutative algebras arise naturally in the study of dynamics on abelian varieties. Dynamics is the study of all iterations of a given self-map  $f : X \rightarrow X$  on a certain space  $X$ . If  $X$  is an abelian variety with an origin  $O$ , then every self-map of  $X$  is a composition of a translation and an endomorphism, namely, a self-map that fixes  $O$ . Endomorphisms of  $X$  form an algebra that is typically noncommutative, usually denoted  $\text{End}(X)$ . If  $f : X \rightarrow X$  is an endomorphism, then the set  $\{f^n : n \geq 1\}$  of all iterates of  $f$  is a semigroup of the multiplicative group of the endomorphism algebra. Many properties about an iterate of  $f$  can be reduced to an equation on  $\text{End}(X)$ ; dynamics problems related to such properties can thus be reduced to the study of solutions in a certain semigroup to an equation on  $\text{End}(X)$ . It turns out that an orbit intersection problem on



an abelian variety can be reduced to a unit equation on its endomorphism algebra involving semigroups  $\Gamma_1, \Gamma_2$ , each generated by one element. In particular, Theorem I.3.1 implies the following result, which is part of the initial motivation for our study of Theorem I.3.1.

**Corollary I.3.2** (Theorem II.1.6). *Let  $E$  be an elliptic curve over an algebraically closed field  $k$ , and let  $f, g : E \rightarrow E$  be regular maps of degrees greater than 1. If there are points  $A, B \in E(k)$  such that the forward orbits  $O_f(A) := \{A, f(A), f^2(A), \dots\}$  and  $O_g(B) := \{B, g(B), g^2(B), \dots\}$  have infinite intersection, then  $f$  and  $g$  have a common iterate, namely,  $f^{m_0} = g^{n_0}$  for some positive integers  $m_0, n_0$ . The result is effective in a sense to be detailed in II.1.6.*

Analogous results have been proven for various other varieties  $E$  in characteristic zero. The innovative case of Corollary I.3.2 is when  $E$  is a supersingular elliptic curve in positive characteristic, in which case  $\text{End}(E)$  is noncommutative and lies in a quaternion algebra. In Chapter II, we will discuss a natural generalization of Theorem I.3.2 and a further problem about unit equations that it motivates.

### I.3.2: Chapter III: Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity

Let  $\mathbb{F}_q$  be the finite field of  $q$  elements, and let  $\text{Mat}_n(\mathbb{F}_q)$  denote the set of  $n$  by  $n$  matrices over  $\mathbb{F}_q$ . We use the  $q$ -Pochhammer symbols

$$(a; t)_n := (1 - a)(1 - ta) \dots (1 - t^{n-1}a), \quad (\text{I.3.2})$$

and

$$(a; t)_\infty := (1 - a)(1 - ta)(1 - t^2a) \dots \quad (\text{I.3.3})$$

For a fixed  $t$  with  $|t| < 1$ , the infinite product defining  $(a; t)_\infty$  converges for all  $a \in \mathbb{C}$ , and it defines a power series in  $a$ .

Our main theorem in Chapter III counts pairs of matrices that annihilate each other, by giving a generating function.

**Theorem I.3.3** (Theorem III.1.2). *Let  $q > 1$  be a prime power. We have the following identity of power series in  $x$ :*

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \text{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \frac{1}{(x; q^{-1})_\infty^2} H_q(x), \quad (\text{I.3.4})$$

where  $|\mathrm{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$ , and the power series  $H_q(x)$  is an entire function defined for all  $x \in \mathbb{C}$ . Moreover, we have a formula for  $H_q(x)$  given by

$$H_q(x) := \sum_{k=0}^{\infty} \frac{q^{-k^2} x^{2k}}{(q^{-1}; q^{-1})_k} (xq^{-k-1}; q^{-1})_{\infty}. \quad (\text{I.3.5})$$

The significance of this result should be viewed in the context of a more general counting question. Let  $f_1, \dots, f_r$  be polynomials in  $\mathbb{F}_q[t_1, \dots, t_m]$ . Consider the set

$$M_n := \left\{ (A_1, \dots, A_m) \left| \begin{array}{l} A_i \in \mathrm{Mat}_n(\mathbb{F}_q), A_i A_j = A_j A_i \\ f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \leq s \leq r \end{array} \right. \right\}. \quad (\text{I.3.6})$$

It turns out that the count  $|M_n|$  is determined by the isomorphism class of the affine variety  $X := \mathrm{Spec} \mathbb{F}_q[t_1, \dots, t_m]/(f_1, \dots, f_r)$ . We denote by  $\widehat{Z}_X(x)$  the generating function

$$\widehat{Z}_X(x) := \sum_{n=0}^{\infty} \frac{|M_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n. \quad (\text{I.3.7})$$

Many classical results about matrix enumerations can be restated as the computation of  $\widehat{Z}_X(x)$  for certain examples of  $X$ . For example, the left-hand side of Feit and Fine's result [FF60] counting pairs of commuting matrices is precisely  $\widehat{Z}_X(x)$  where  $X$  is the affine plane:

$$\sum_{n=0}^{\infty} \frac{|\{A, B \in \mathrm{Mat}_n(\mathbb{F}_q) : AB = BA\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i,j \geq 1} \frac{1}{1 - x^i q^{2-j}}. \quad (\text{I.3.8})$$

Our work can be viewed as an analogue of Feit and Fine's result (while Chapter IV is a generalization in another direction). The advantage of this formulation is that  $\widehat{Z}_X(x)$  is determined by the geometry of  $X$  in a strong sense. In fact, if  $X$  is a smooth curve or a smooth surface over  $\mathbb{F}_q$ , then we have the following formulas for  $\widehat{Z}_X(x)$  in terms of the (often well-known) Hasse–Weil zeta function of  $X$ :

**Proposition I.3.4** (Proposition III.4.5). *Let  $Z_X(x)$  be the Hasse–Weil zeta series of  $X$ . Then*

(a) *If  $X$  is a smooth curve over  $\mathbb{F}_q$ , then*

$$\widehat{Z}_X(x) = \prod_{i=1}^{\infty} Z_X(q^{-i}x) \in \mathbb{C}[[x]]. \quad (\text{I.3.9})$$

(b) If  $X$  is a smooth surface over  $\mathbb{F}_q$ , then

$$\widehat{Z}_X(x) = \prod_{i,j \geq 1} Z_X(q^{-j}x^i) \in \mathbb{C}[[x]]. \quad (\text{I.3.10})$$

For general  $X$ , the generating function  $\widehat{Z}_X(x)$  is determined by the local geometry of  $X$  at each point, via an Euler product formula (Proposition III.4.2). In particular, if  $X$  is a singular curve, then to compute  $\widehat{Z}_X(x)$ , it suffices to compute a related generating function (called the *local Cohen–Lenstra series* in Chapter III) attached to each singularity of  $X$ . Therefore, the study of such series attached to each curve singularity plays a central role in the study of  $\widehat{Z}_X(x)$  for singular curves. Theorem I.3.3 essentially computes the local Cohen–Lenstra series attached to the *nodal* singularity; in fact, the left-hand side of the identity in Theorem I.3.3 is precisely  $\widehat{Z}_{X_0}(x)$  where  $X_0 = \{(u, v) : uv = 0\}$  is the union of coordinate axes on a plane, which has a node at the origin.

Our main result (Theorem I.3.3) gives the first case where  $\widehat{Z}_X(x)$  is computed and  $X$  is singular. The case for other types of singularities is largely unknown, and  $\widehat{Z}_X(x)$  is not expected to have an explicit formula as simple as Proposition I.3.4. However, Theorem I.3.3 exhibits a special feature that is potentially true for curves with other types of singularities. Theorem I.3.3 gives a factorization of  $\widehat{Z}_{X_0}(x)$  into a simple infinite product and a mysterious holomorphic factor. This factorization implies that  $\widehat{Z}_{X_0}(x)$ , a power series that is convergent on  $|x| < 1$ , turns out surprisingly to have a meromorphic continuation to all of  $\mathbb{C}$ . In Question III.1.3 and its following discussions, we will give some geometric heuristics in attempt to explain the factorization.

Finally, we discuss why our result is essentially a counting problem over a moduli space. Giving a pair  $(A, B)$  of mutually annihilating commuting  $n$  by  $n$  matrices is equivalent to giving the vector space  $\mathbb{F}_q^n$  a structure as an  $\mathbb{F}_q[u, v]/(uv)$ -module, by requiring that  $u$  acts on  $\mathbb{F}_q^n$  as  $A$  and  $v$  as  $B$ . In other words, the set of pairs of mutually annihilating matrices parametrizes finite-dimensional modules over  $\mathbb{F}_q[u, v]/(uv)$ , in which sense it is viewed as a moduli space of finite-dimensional modules over  $\mathbb{F}_q[u, v]/(uv)$ . Geometrically, a finite-dimensional module over  $\mathbb{F}_q[u, v]/(uv)$  is a finite-length coherent sheaf over the affine variety  $X_0 = \text{Spec } \mathbb{F}_q[u, v]/(uv)$ . Therefore, the generating function in Theorem I.3.3 counts finite-length coherent sheaves over  $X_0$ . The discussion above applies to any affine variety  $X$ , and in fact  $\widehat{Z}_X(x)$  counts finite-length coherent sheaves over  $X$ . More precisely,

$$\widehat{Z}_X(x) = \sum_M \frac{1}{|\text{Aut } M|} x^{\dim_{\mathbb{F}_q} H^0(X; M)}, \quad (\text{I.3.11})$$

where  $M$  ranges over isomorphism classes of finite-length coherent sheaves over  $X$ , and  $H^0(X; M)$  is the space of global sections of  $M$ . We note the factor  $1/|\text{Aut } M|$  from the above expression, which means that the said count is with respect to a weight inversely proportional to the size of the automorphism group. This weighting was originally due to Cohen and Lenstra in their work [CL84] about the statistics of random abelian groups.

### I.3.3: Chapter IV: Counting on the variety of modules over the quantum plane

Fix a nonzero element  $\zeta$  in  $\mathbb{F}_q$ , the finite field with  $q$  elements. Let  $\text{ord}(\zeta)$  denote the smallest positive integer  $m$  such that  $\zeta^m = 1$  in  $\mathbb{F}_q$ . The goal of this chapter is to count the cardinality of the following set

$$K_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}, \quad (\text{I.3.12})$$

namely, to count the number of solutions  $(A, B)$  of the matrix equation  $AB = \zeta BA$  where  $A, B$  are  $n$  by  $n$  matrices over  $\mathbb{F}_q$ .

As the main result, we show that the above count can be given by the following formula in terms of an identity of power series in  $\mathbb{Q}[[x]]$ :

**Theorem I.3.5.**

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} F_{\text{ord}(\zeta)}(x^i; q), \quad (\text{I.3.13})$$

where

$$F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \dots} \quad (\text{I.3.14})$$

and  $|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$  is the number of nonsingular  $n$  by  $n$  matrices over  $\mathbb{F}_q$ .

In particular, the count only depends on the order of  $\zeta$  as a root of unity.

We also show some factorization identities between generating functions of several counts related to  $|K_{\zeta,n}(\mathbb{F}_q)|$ , best summarized as “Mat = GL  $\oplus$  Nilp”. Let  $\text{Nilp}_n(\mathbb{F}_q)$  denote the set of  $n$  by  $n$  nilpotent matrices over  $\mathbb{F}_q$ . For any combination  $(F, G)$  where each of  $F$  and  $G$  is one of the symbols Mat, GL or Nilp, we define

$$K_{\zeta,n}^{F \times G} := \{(A, B) \in F_n(\mathbb{F}_q) \times G_n(\mathbb{F}_q) : AB = \zeta BA\}. \quad (\text{I.3.15})$$

We define the generating function  $E_{\zeta}^{F \times G}(x; q)$  as

$$E_{\zeta}^{F \times G}(x; q) := \sum_{n=0}^{\infty} \frac{|K_{\zeta,n}^{F \times G}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n. \quad (\text{I.3.16})$$

For example,  $E_\zeta^{\text{Mat} \times \text{Mat}}(x; q)$  is the left-hand side of (I.3.13). Our observations are the identities

$$E_\zeta^{\text{Mat} \times \text{Mat}}(x; q) = E_\zeta^{\text{Mat} \times \text{GL}}(x; q)E_\zeta^{\text{Mat} \times \text{Nilp}}(x; q); \quad (\text{I.3.17})$$

$$E_\zeta^{\text{GL} \times \text{Mat}}(x; q) = E_\zeta^{\text{GL} \times \text{GL}}(x; q)E_\zeta^{\text{GL} \times \text{Nilp}}(x; q); \quad (\text{I.3.18})$$

$$E_\zeta^{\text{Nilp} \times \text{Mat}}(x; q) = E_\zeta^{\text{Nilp} \times \text{GL}}(x; q)E_\zeta^{\text{Nilp} \times \text{Nilp}}(x; q). \quad (\text{I.3.19})$$

We also compute each of the generating functions above, from which the above identities are evident. It turns out that each generating function  $E_\zeta^{\text{F} \times \text{G}}(x; q)$  only depends on the order of  $\zeta$  as a root of unity, and is of the form

$$E_\zeta^{\text{F} \times \text{G}}(x; q) = \prod_{i=1}^{\infty} e_{\text{ord}(\zeta)}^{\text{F} \times \text{G}}(x^i; q), \quad (\text{I.3.20})$$

for some power series  $e_m^{\text{F} \times \text{G}}(x; q)$  in  $x$ . We have

$$e_m^{\text{Mat} \times \text{Mat}}(x; q) = \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \dots} \quad (\text{Theorem I.3.5}) \quad (\text{I.3.21})$$

$$e_m^{\text{Mat} \times \text{GL}}(x; q) = \frac{1 - x^m}{(1 - x)(1 - x^m q)} \quad (\text{I.3.22})$$

$$e_m^{\text{Mat} \times \text{Nilp}}(x; q) = \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \dots} \quad (\text{I.3.23})$$

$$e_m^{\text{GL} \times \text{GL}}(x; q) = \frac{1 - x^m}{1 - x^m q} \quad (\text{I.3.24})$$

$$e_m^{\text{GL} \times \text{Nilp}}(x; q) = \frac{1}{1 - x} \quad (\text{I.3.25})$$

$$e_m^{\text{Nilp} \times \text{Nilp}}(x; q) = \frac{1}{(1 - xq^{-1})(1 - xq^{-2}) \dots} \quad (\text{I.3.26})$$

One may verify the identities (I.3.17) through (I.3.19) by verifying the corresponding identities in the list above. We point out an interesting observation that  $e_m^{\text{F} \times \text{G}}(x; q)$  does not depend on  $m$  as long as either  $F$  or  $G$  is Nilp.

Similar to Chapter III, the set  $K_{\zeta, n}(\mathbb{F}_q)$  in Theorem I.3.5 can be viewed as a moduli space of modules, and the coefficients of the generating function  $E_\zeta^{\text{Mat} \times \text{Mat}}(x; q)$  in Theorem I.3.5 are the weighted counts of such modules following the Cohen–Lenstra measure. In specific,  $K_{\zeta, n}(\mathbb{F}_q)$  parametrizes  $n$ -dimensional modules over the algebra  $\mathbb{F}_q\{X, Y\}/(XY - \zeta YX)$  generated by noncommutative generators  $X$  and  $Y$  with relation  $XY = \zeta YX$ . This algebra is called the *quantum plane* and has been extensively studied (see for instance,

[CL19]) as an important example of quantum deformation. Moreover, we have

$$E_{\zeta}^{\text{Mat} \times \text{Mat}}(x; q) = \sum_M \frac{1}{|\text{Aut } M|} x^{\dim_{\mathbb{F}_q} M}, \quad (\text{I.3.27})$$

where  $M$  ranges over finite-dimensional modules over the quantum plane  $\mathbb{F}_q\{X, Y\}/(XY - \zeta YX)$ .

### I.3.4: Interlude about motivicity

The common theme of Chapter V and Chapter VI is finding analogues of point-counting statements of configuration spaces. Given a space  $X$  (where a space could refer to a topological space or a quasi-projective variety), consider the *ordered configuration space*

$$F(X, n) := \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ for all } i \neq j\}, \quad (\text{I.3.28})$$

and define the (*unordered*) *configuration space* as a topological or scheme-theoretic quotient

$$\text{Conf}^n(X) := F(X, n)/S_n \quad (\text{I.3.29})$$

by the symmetric group  $S_n$ . As an example notable in number theory, if  $X$  is the affine line  $\mathbb{A}^1$  over the finite field  $\mathbb{F}_q$ , then the set of  $\mathbb{F}_q$ -points of  $\text{Conf}^n(\mathbb{A}^1)$  can be identified as the set of monic square-free polynomials of degree  $n$  over  $\mathbb{F}_q$ . The point counting for any configuration space is well-known [VW15, Proposition 5.9]: for any quasiprojective variety  $X$  over  $\mathbb{F}_q$ , the numbers of  $\mathbb{F}_q$ -points of configuration spaces of  $X$  satisfy

$$\sum_{n=0}^{\infty} |\text{Conf}^n(X)(\mathbb{F}_q)| t^n = Z_X(t)/Z_X(t^2), \quad (\text{I.3.30})$$

where  $Z_X(t)$  is the (well-known) Hasse–Weil zeta function of  $X$ . In particular, applying this formula to  $X = \mathbb{A}^1$ , we immediately recover the classical fact that the number of monic square-free polynomials of degree  $n$  over  $\mathbb{F}_q$  is  $q^n - q^{n-1}$  (except when  $n = 0, 1$ , in which cases the number is  $q^n$ ).

The point count identity holds in a stronger sense: the point count of  $\text{Conf}^n(X)$  is known not by coincidence, but because it is known how  $\text{Conf}^n(X)$  is made up of other better-understood varieties by taking disjoint unions and complements. As an illustrating example, the projective space  $\mathbb{P}^n$  over  $\mathbb{F}_q$  has  $1 + q + \dots + q^n$  points, but moreover, the projective space  $\mathbb{P}^n$  can be decomposed as a disjoint union  $\mathbb{P}^n = \mathbb{A}^0 \cup \mathbb{A}^1 \cup \dots \cup \mathbb{A}^n$ , which means that the point count formula  $|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \dots + q^n$  really stems from a stronger geometric

statement. To formulate this notion, we consider the Grothendieck groups  $K_0(\text{Var}_{\mathbb{F}_q})$  and  $K_0(\text{Var}_{\mathbb{C}})$  of  $\mathbb{F}_q$ - and complex varieties, defined as the abelian group generated by  $[X]$  for all isomorphism classes of varieties  $X$ , with relation

$$[X] = [Z] + [X \setminus Z] \tag{I.3.31}$$

whenever  $Z$  is a closed subvariety of  $X$ . The Grothendieck group allows us to talk about statements analogous to point counting for complex varieties, where the notion of point counting is not available. The original [VW15, Proposition 5.9] by Vakil and Wood is in fact a formula for  $[\text{Conf}^n(X)]$  in the Grothendieck group, where  $X$  is a variety over any field.

For a variety  $X$  over any field, we call  $[X]$  the *motivic class* of  $X$  in the Grothendieck group, and we call an identity in the Grothendieck group to be a *motivic identity*. An invariant attached to a variety is called a *motivic invariant* if it only depends on the motivic class of the variety. For example, the point count is a motivic invariant of  $\mathbb{F}_q$ -varieties, and the Euler characteristic (an important topological invariant) is a motivic invariant of complex varieties. A motivic identity always implies an identity of a motivic invariant, and very often a point count identity for a good geometric reason can be upgraded to a motivic identity. For example, we have

$$|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1}), \tag{I.3.32}$$

and in fact it is also true that

$$[\text{GL}_n(\mathbb{C})] = ([\mathbb{C}^n] - [\mathbb{C}^0])([\mathbb{C}^n] - [\mathbb{C}^{n-1}]) \dots ([\mathbb{C}^n] - [\mathbb{C}^{n-1}]) \tag{I.3.33}$$

in the Grothendieck ring of complex varieties. (The Grothendieck group has a ring structure given by the Cartesian product; hence, the Grothendieck group is also called the Grothendieck ring.) As another example, Bryan and Morrison [BM15] refined the Feit–Fine formula about pairs of commuting matrices (I.3.8) into a motivic identity in the ring  $K_0(\text{Var}_{\mathbb{C}})[[\mathbb{L}^{-1}]]$ , where  $\mathbb{L} := [\mathbb{C}]$ :

$$\sum_{n=0}^{\infty} \frac{[\{A, B \in \text{Mat}_n(\mathbb{C}) : AB = BA\}]}{[\text{GL}_n(\mathbb{C})]} = \prod_{i,j \geq 1} \frac{1}{1 - \mathbb{L}^{2-j} x^i}. \tag{I.3.34}$$

We also point out that all of the main results of Chapter III and Chapter IV are in fact motivic, in the sense that the motivic identities obtained from replacing  $q$  in the counting formula with  $\mathbb{L}$  hold.

Chapters V and VI focus on the Betti numbers and the mixed Hodge numbers of config-

uration spaces of complex varieties. The Betti number is a topological invariant, while the mixed Hodge number is an invariant that depends on the structure as a complex variety. Nevertheless, mixed Hodge numbers always refine the data of Betti numbers. Neither the Betti number nor the mixed Hodge number is an motivic invariant. However, it turns out that motivic identities of configuration spaces tend to have analogues about Betti numbers and mixed Hodge numbers in favorable situations. Chapters V and VI exhibit two instances of such analogues of motivic identities. In this sense, the main results of these chapters are said to be about “motivic aspects” of the configuration spaces. These results require separate proofs and are not consequences of the motivic formula for configuration spaces by Vakil and Wood, though the main result of Chapter V takes advantage of the motivic formula in an important way.

### I.3.5: Chapter V: Rationality for the Betti numbers of the unordered configuration spaces of a punctured torus (joint work with G. Cheong)

Let  $E$  be an elliptic curve over  $\mathbb{C}$ , and  $E^\times$  an open subset of  $E$  obtained by removing one point. Topologically,  $E$  is a torus. For any complex variety  $X$ , we consider the Betti number  $h^i(X)$  for each  $i \geq 0$  and the mixed Hodge number  $h^{p,q;i}(X)$  (from Deligne’s mixed Hodge theory, see for instance [Del71]) for each tuple of  $p, q, i \geq 0$ . We set  $h^i(X) = 0$  if  $i < 0$ , and  $h^{p,q;i}(X) = 0$  if at least one of  $p, q, i$  is negative. The mixed Hodge numbers refine the Betti numbers by

$$h^i(X) = \sum_{p,q \geq 0} h^{p,q;i}(X). \quad (\text{I.3.35})$$

We show that the Betti and mixed Hodge numbers of configuration spaces of  $E^\times$  are given by rational generating functions, but with a special degree shifting.

**Theorem I.3.6.** *Keeping the notation as above, we have*

$$\sum_{i,n \geq 0} (-1)^i h^i(\text{Conf}^n(E^\times)) u^{2n-w(i)} t^n = \frac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2} \quad (\text{I.3.36})$$

and

$$\sum_{p,q,i,n \geq 0} (-1)^i h^{n-p,n-q;i}(\text{Conf}^n(E^\times)) x^p y^q u^{2n-w(i)} t^n = \frac{(1-xut)(1-yut)(1-xyu^2t^2)}{(1-xyu^2t)(1-xut^2)(1-yut^2)}, \quad (\text{I.3.37})$$

where  $w(i) = \lfloor 3i/2 \rfloor$ . The above formulas determine all Betti and mixed Hodge numbers of  $\text{Conf}^n(E^\times)$  because  $w : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  is strictly increasing.



The key feature of our result is that the rational functions that appear on the right-hand sides of the above identities are motivic invariants of configuration spaces of  $E^\times$ . These motivic invariants are ultimately derived from the following invariant, whose definition for a connected smooth  $d$ -dimensional complex variety  $X$  reads

$$h_{\text{vir}}^{p,q}(X) := \sum_{i=0}^{\infty} (-1)^i h^{d-p, d-q; i}(X) \quad (\text{I.3.38})$$

for each fixed  $p, q \geq 0$ . This is a motivic invariant, which implies that  $h_{\text{vir}}^{p,q}(\text{Conf}^n(E^\times))$  is determined by the motivic formula for  $[\text{Conf}^n(E^\times)]$  given by Vakil and Wood. However, the invariant  $h_{\text{vir}}^{p,q}$  does not recover any Betti number or mixed Hodge number in general, because it only records an alternating sum of mixed Hodge numbers.

The main novelty of our result is that the motivic formula for  $[\text{Conf}^n(E^\times)]$  turns out to recover all Betti and mixed Hodge numbers of  $\text{Conf}^n(E^\times)$ . We prove a geometric statement specific to the punctured elliptic curve  $E^\times$ :

**Theorem I.3.7.** *For any  $i, n \geq 0$ , the mixed Hodge structure of the  $i$ -th cohomology group  $H^i(\text{Conf}^n(E^\times); \mathbb{Q})$  of  $\text{Conf}^n(E^\times)$  is pure of weight  $w(i) = \lfloor 3i/2 \rfloor$ , which means  $h^{p,q; i}(\text{Conf}^n(E^\times)) = 0$  unless  $p + q = w(i)$ .*

Given Theorem I.3.7, for each  $p, q \geq 0$ , the summation  $\sum_{i=0}^{\infty} (-1)^i h^{d-p, d-q; i}(\text{Conf}^n(E^\times))$  has at most one nonzero term, so  $h_{\text{vir}}^{p,q}(\text{Conf}^n(E^\times))$  recovers all mixed Hodge numbers of  $\text{Conf}^n(E^\times)$ . The weight  $w(i)$  that appears in Theorem I.3.7 is the geometric significance of the degree shifting in Theorem I.3.6.

The Betti numbers of  $\text{Conf}^n(E^\times)$  are known by [DCK17]. We note that the Betti number is a topological invariant that does not require the mixed Hodge theory, but our explanation of the rational generating function for Betti numbers crucially requires Theorem I.3.7, a mixed-Hodge-theoretic statement. Thus, the Betti number statement of Theorem I.3.6 can be viewed as an application of Deligne's mixed Hodge theory.

### I.3.6: Chapter VI: Cohomology of configuration spaces on punctured varieties

Consider a connected smooth noncompact complex variety  $X$ , and let  $X^\times$  be an open subset of  $X$  obtained from removing one point. The main result of Chapter VI is that the Betti (mixed Hodge) numbers of configuration spaces of  $X^\times$  are determined by the Betti (mixed Hodge) numbers of configuration spaces of  $X$ , under a flexible assumption for  $X$ . For brevity, we only state a special case where the assumption for  $X$  is satisfied.

**Theorem I.3.8.** *Let  $X$  be a connected compact smooth complex variety of dimension  $d$  with  $r \geq 1$  points punctured (in particular,  $X$  is never compact). Let  $X^\times$  be a one-puncture of*

$X$ , then we have

$$\sum_{i,n \geq 0} h^i(\mathrm{Conf}^n(X^\times))(-u)^i t^n = \frac{1}{1+u^{2d-1}t} \sum_{i,n \geq 0} h^i(\mathrm{Conf}^n(X))(-u)^i t^n \quad (\text{I.3.39})$$

and

$$\begin{aligned} & \sum_{p,q,i,n \geq 0} h^{p,q;i}(\mathrm{Conf}^n(X^\times)) x^p y^q (-u)^i t^n \\ &= \frac{1}{1+(xy)^d u^{2d-1} t} \sum_{p,q,i,n \geq 0} h^{p,q;i}(\mathrm{Conf}^n(X)) x^p y^q (-u)^i t^n. \end{aligned} \quad (\text{I.3.40})$$

The Betti number formula (I.3.39) is actually a result of Kallel [Kal08, Theorem 1.5]. One main contribution of our work is explaining the connection between (I.3.39) and the Vakil–Wood motivic formula for configuration spaces using the mixed Hodge number formula (I.3.40); this connection will be detailed below. Another main contribution is providing a different approach to (I.3.39), which has yielded further generalizations and refinements of Theorem I.3.8; see Chapter VI for the precise statements of the generalizations and refinements.

Since removing a point has a straightforward effect to the motivic class (namely,  $[X^\times] = [X] - [\mathrm{pt}]$  in the Grothendieck ring of varieties), the motivic formula of Vakil and Wood would imply a simple relation between the motivic classes of  $\mathrm{Conf}^n(X^\times)$  and the motivic classes of  $\mathrm{Conf}^n(X)$ :

$$\sum_{n=0}^{\infty} [\mathrm{Conf}^n(X^\times)] t^n = \frac{1}{1+t} \sum_{n=0}^{\infty} [\mathrm{Conf}^n(X)] t^n \quad (\text{I.3.41})$$

for *any* variety  $X$ .

Specializing the above formula to the mixed-Hodge-theoretic motivic invariant introduced in (I.3.38), we get an identity

$$\begin{aligned} & \sum_{p,q,n \geq 0} \left( \sum_{i=0}^{\infty} (-1)^i h^{dn-p, dn-q; i}(\mathrm{Conf}^n(X^\times)) \right) x^p y^q t^n \\ &= \frac{1}{1+t} \sum_{p,q,n \geq 0} \left( \sum_{i=0}^{\infty} (-1)^i h^{dn-p, dn-q; i}(\mathrm{Conf}^n(X)) \right) x^p y^q t^n \end{aligned} \quad (\text{I.3.42})$$

where  $X$  is any connected smooth  $d$ -dimensional complex variety. We note that (I.3.42) implies neither (I.3.39) nor (I.3.40), because each coefficient of (I.3.42) is an alternating sum of mixed Hodge numbers and does not recover each mixed Hodge number individually.

However, (I.3.40) implies both (I.3.39) and (I.3.42) by careful substitutions of the variables  $x, y, u, t$ . In conclusion, our new result (I.3.40) about mixed Hodge numbers is a common refinement of Kallel's (I.3.39) about Betti numbers and the identity (I.3.42) of motivic invariants, thus explaining the connection between (I.3.39) and (I.3.42).

# CHAPTER II

## Unit Equations on Quaternions

The content of this chapter is published in [Hua20b].

### Abstract

A classical result about unit equations says that if  $\Gamma_1$  and  $\Gamma_2$  are finitely generated subgroups of  $\mathbb{C}^\times$ , then the equation  $x + y = 1$  has only finitely many solutions with  $x \in \Gamma_1$  and  $y \in \Gamma_2$ . We study a noncommutative analogue of the result, where  $\Gamma_1, \Gamma_2$  are finitely generated subsemigroups of the multiplicative group of a quaternion algebra. We prove an analogous conclusion when both semigroups are generated by algebraic quaternions with norms greater than 1 and one of the semigroups is commutative. As an application in dynamics, we prove that if  $f$  and  $g$  are endomorphisms of a curve  $C$  of genus 1 over an algebraically closed field  $k$ , and  $\deg(f), \deg(g) \geq 2$ , then  $f$  and  $g$  have a common iterate if and only if some forward orbit of  $f$  on  $C(k)$  has infinite intersection with an orbit of  $g$ .

### II.1: Introduction

A classical result about unit equations states that the equation  $f + g = 1$  has only finitely many solutions in a given finitely generated semigroup  $\Gamma$  in  $K^\times$ , where  $K$  is a field of characteristic zero. Unit equations have had important applications in many areas of mathematics, including Diophantine geometry ([HS13, Lan60]), arithmetic dynamics [EG15, p. 291] and variants of the Mordell–Lang conjecture (for instance, see [EG15, p. 321]). Extensions of the classical result have also been studied, for example, see [KP17, Vol98] in the characteristic  $p$  setting.

In this paper we present a class of semigroups in the standard quaternion algebra over  $\mathbb{R}$  for which the finiteness of solutions of the unit equation holds. This is the first analogous result in the noncommutative setting. In light of the many applications of unit equations, this raises the intriguing possibility that some of those applications might have noncommutative

analogues.

Let  $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$  denote the quaternion algebra  $\mathbb{H}$  over  $\mathbb{R}$ , with the standard multiplication law  $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ . For an element  $\alpha = a + bi + cj + dk \in \mathbb{H}$ , where  $a, b, c, d \in \mathbb{R}$ , define its conjugation to be  $\bar{\alpha} = a - bi - cj - dk$ , its norm to be  $N(\alpha) = \alpha\bar{\alpha} = \bar{\alpha}\alpha = a^2 + b^2 + c^2 + d^2$ , and its trace  $tr(\alpha) = \alpha + \bar{\alpha} = 2a$ . Write  $|\alpha| = \sqrt{N(\alpha)}$ .

We say that a quaternion  $\alpha = a + bi + cj + dk \in \mathbb{H}$  is *algebraic* if all coordinates  $a, b, c, d$  are algebraic over  $\mathbb{Q}$ . This is equivalent to requiring that  $\alpha$  satisfies a polynomial equation with coefficients in  $\mathbb{Q}$ , or that  $\mathbb{Q}[\alpha]$  is a finite field extension of  $\mathbb{Q}$ . Indeed,  $\alpha$  always satisfies the quadratic equation

$$X^2 - tr(\alpha)X + N(\alpha) = 0 \tag{II.1.1}$$

and if  $a, b, c, d \in \overline{\mathbb{Q}}$ , then so are  $tr(\alpha)$  and  $N(\alpha)$ .

Denote by  $\mathbb{H}_a$  the subalgebra of all quaternions that are algebraic.

**Theorem II.1.1.** *Let  $\Gamma_1, \Gamma_2$  be semigroups of  $\mathbb{H}_a^\times$  generated by finitely many elements of norms greater than 1, and fix  $a, a', b, b' \in \mathbb{H}_a^\times$ . If  $\Gamma_1$  is commutative, then the equation*

$$afa' + bgb' = 1 \tag{II.1.2}$$

*has only finitely many solutions with  $f \in \Gamma_1$  and  $g \in \Gamma_2$ . In fact, for solutions  $(f, g)$ , we have effectively computable upper bounds for  $|f|, |g|$  that depend only on  $a, a', b, b'$  and generators of  $\Gamma_1, \Gamma_2$ .*

We emphasize that even though  $\Gamma_1$  is commutative, the semigroup  $\Gamma_2$  need not be commutative, and that  $a, a'$  and  $\Gamma_1$  typically will not commute with each other. The proof relies on the following result, which implies that if a certain quaternion unit equation has infinitely many solutions, then so does another equation of a different type. We note that Theorem II.1.2 applies in greater generality than Theorem II.1.1, as Theorem II.1.2 does not require  $\Gamma_1$  to be commutative.

**Theorem II.1.2.** *Let  $\Gamma_1, \Gamma_2$  be semigroups of  $\mathbb{H}_a^\times$  generated by finitely many elements of norms greater than 1, and fix  $a, a', b, b' \in \mathbb{H}_a^\times$ . Then the equation*

$$afa' + bgb' = 1 \tag{II.1.3}$$

*has only finitely many solutions with  $f \in \Gamma_1$  and  $g \in \Gamma_2$  such that  $|1 - afa'| \neq |afa'|$ . In fact, for such pairs  $(f, g)$ , we have effectively computable upper bounds for  $|f|, |g|$  that depend only on  $a, a', b, b'$  and generators of  $\Gamma_1, \Gamma_2$ .*

Given Theorem II.1.2, in order to prove Theorem II.1.1 it suffices to prove the next result which involves only the semigroup  $\Gamma_1$ :

**Theorem II.1.3.** *Let  $\Gamma$  be a semigroup generated by finitely many elements in  $\mathbb{H}_a$  with norms greater than 1, and fix  $a, a' \in \mathbb{H}_a^\times$ . If  $\Gamma$  is commutative, then the equation*

$$|1 - afa'| = |afa'| \tag{II.1.4}$$

*has only finitely many solutions with  $f \in \Gamma$ . In fact, for solutions  $f \in \Gamma$ , we have an effectively computable upper bound for  $|f|$  that depends only on  $a, a'$  and generators of  $\Gamma$ .*

We remark that Theorem II.1.3 is the only step in the proof of Theorem II.1.1 that uses the commutativity of  $\Gamma_1$ , so any generalization of Theorem II.1.3 would immediately yield a generalization of Theorem II.1.1.

In light of the above results, we make the following conjecture about noncommutative unit equations:

**Conjecture II.1.4.** Let  $\Gamma_1, \Gamma_2$  be finitely generated semigroups of the multiplicative group  $A^\times$  of a finite dimensional division algebra  $A$  over  $\mathbb{Q}$ . Then for any fixed  $a, a', b, b' \in A^\times$ , the unit equation  $afa' + bgb' = 1$  has only finitely many solutions with  $f \in \Gamma_1$  and  $g \in \Gamma_2$ .

Moreover, there is an effectively computable finite subset  $S \subseteq \Gamma_1 \times \Gamma_2$  in terms of  $a, a', b, b'$  and generators of  $\Gamma_1, \Gamma_2$ , such that all solutions  $(f, g) \in \Gamma_1 \times \Gamma_2$  must lie in  $S$ .

The referee kindly points out that the ineffective part of the conjecture is true in the case where all the semigroups are commutative:

**Proposition II.1.5.** *Let  $k$  be a field of characteristic zero and  $A$  be a finite-dimensional division algebra over  $k$ . Let  $\Gamma_1, \dots, \Gamma_m$  be abelian and finitely generated subgroups of the multiplicative group  $A^\times$ . Then for any fixed  $a_1, \dots, a_m, b_1, \dots, b_m \in A^\times$ , the unit equation*

$$a_1 f_1 b_1 + \dots + a_m f_m b_m = 1 \tag{II.1.5}$$

*has only finitely many nondegenerate solutions  $(f_1, \dots, f_m) \in \Gamma_1 \times \dots \times \Gamma_m$ , i.e., solutions such that no proper subsum equals 1.*

This, of course, implies the case where all  $\Gamma_i$  are *virtually* abelian, in the sense that  $\Gamma_i$  has a finite-index abelian subgroup. Moreover, as is mentioned by the referee, every semigroup that does not contain a free semigroup of rank two is contained in a virtually abelian subgroup of  $A^\times$  (see Proposition II.7.2), so Proposition II.1.5 also holds if  $\Gamma_i$  does not contain a free semigroup of rank two. We emphasize that Theorem II.1.1 is the only

currently proven case of Conjecture [II.1.4](#) where some of the semigroups are not contained in virtually abelian subgroups of  $A^\times$ . It is also the only known case where  $A$  is noncommutative and one knows an effectively computable finite set that contains all the solutions.

In Section [II.6](#), we will discuss a possible  $p$ -adic approach to Conjecture [II.1.4](#), and will give a counterexample to the matrix algebra analogue of Conjecture [II.1.4](#) in Example [II.6.1](#).

Our main theorem has the following consequence about intersections of orbits of endomorphisms of a genus-1 curve in arbitrary characteristic.

**Corollary II.1.6.** *Let  $E$  be an elliptic curve over an algebraically closed field  $k$ , and let  $f, g : E \rightarrow E$  be regular maps of degrees greater than 1. If there are points  $A, B \in E(k)$  such that the forward orbits  $O_f(A) := \{A, f(A), f^2(A), \dots\}$  and  $O_g(B) := \{B, g(B), g^2(B), \dots\}$  have infinite intersection, then  $f$  and  $g$  have a common iterate, namely,  $f^{m_0} = g^{n_0}$  for some positive integers  $m_0, n_0$ .*

*In fact, if  $O_f(A) \cap O_g(B)$  is nonempty, let  $m_0, n_0$  be integers such that  $f^{m_0}(A) = g^{n_0}(B)$ . Then there is an effectively computable constant  $M$  in terms of  $A, B, f, g, m_0, n_0$  such that, if  $f^m(A) = g^n(B)$  for some  $(m, n)$  where either  $m > M$  or  $n > M$ , then  $f^{m_0} = g^{n_0}$ .*

Analogous results have been proven in various cases in characteristic zero, in case  $E$  is replaced by  $\mathbb{A}^1$  [[GTZ12](#)], a linear space [[GN17](#)], or a semiabelian variety [[GN17](#), [GTZ11](#)]. Corollary [II.1.6](#), however, applies to all characteristics.

It would be interesting to study high-dimensional analogues of Corollary [II.1.6](#). For instance, we will show that if certain cases of Conjecture [II.1.4](#) hold, then Corollary [II.1.6](#) remains true if  $E$  is replaced by a *simple* abelian variety, i.e., an abelian variety having no nonzero proper abelian subvarieties. The referee’s Proposition [II.1.5](#) thus yields an unconditional proof of the ineffective part of the simple abelian variety analogue of Corollary [II.1.6](#).

**Corollary II.1.7.** *Let  $X$  be a simple abelian variety over an algebraically closed field  $k$ , and let  $f, g : X \rightarrow X$  be regular maps of degrees greater than 1. If there are points  $A, B \in X(k)$  such that the forward orbits  $O_f(A) := \{A, f(A), f^2(A), \dots\}$  and  $O_g(B) := \{B, g(B), g^2(B), \dots\}$  have infinite intersection, then  $f$  and  $g$  have a common iterate, namely,  $f^{m_0} = g^{n_0}$  for some positive integers  $m_0, n_0$ .*

The characteristic zero case of Corollary [II.1.6](#) and Corollary [II.1.7](#) is an instance of the higher-rank generalization posed in [[GTZ12](#), Question 1.6] of the dynamical Mordell–Lang conjecture [[BGT16](#), Chapter 3]; see also [[GN17](#)]. For positive characteristic, see [[BGT16](#), Chapter 13]. We note that the conclusions of all previous results in characteristic  $p > 0$  involve the more complicated possibility of  $p$ -automatic sequences (e.g., [[Der07](#), [Ghi19](#)]),

whereas the conclusion of Corollary [II.1.6](#) and Corollary [II.1.7](#) is more rigid. This extra possibility also occurs in the positive characteristic version of the original (not dynamical) Mordell–Lang conjecture [[MS04](#)], where it is called an “ $F$ -structure” and where examples are given to show that the possibility cannot be removed.

The rest of the paper is organized as follows. In Section [II.2](#), we state a known Diophantine result. Then Sections [II.3](#), [II.4](#) and [II.5](#) contain proofs of Corollary [II.1.6](#) (together with Corollary [II.1.7](#)), Theorem [II.1.2](#) and Theorem [II.1.3](#), respectively. The proofs are independent of one another, and can be read in any order. Theorem [II.1.1](#) follows immediately from Theorem [II.1.2](#) and Theorem [II.1.3](#). The appendix includes the referee’s proof of Proposition [II.1.5](#) and a result about semigroups not containing noncommutative free semigroups.

## II.2: Linear Forms in Logarithms

The proofs of Theorem [II.1.2](#) and Theorem [II.1.3](#) rely on the following form of Baker’s theorem on Diophantine approximation of logarithms.

**Theorem II.2.1** (Baker, Wüstholz [[BW93](#)]). *Let  $\lambda_1, \dots, \lambda_r$  be complex numbers such that  $e^{\lambda_i}$  are algebraic for  $1 \leq i \leq r$ . Then there are effectively computable constants  $k, C > 0$  depending on  $r$  and  $\lambda_i$  such that*

$$0 < |a_1\lambda_1 + \dots + a_r\lambda_r| \leq kH^{-C} \tag{II.2.1}$$

*has no solutions in  $a_i \in \mathbb{Z}$ , where  $H = \max_{i=1}^r |a_i|$ .*

The effective computability of Theorem [II.2.1](#) implies the effective part of our results, and our proofs will yield explicit bounds in our result if we use an explicit version of Theorem [II.2.1](#) (for example, see [[EG15](#), §3.2]).

## II.3: Proof of Corollary [II.1.6](#) and [II.1.7](#)

In this section, we prove Corollary [II.1.6](#) and a conditional generalization to simple abelian varieties, which implies [II.1.7](#).

*Proof of Corollary [II.1.6](#).* Since  $\deg(f) > 1$ , the regular map  $f$  has a fixed point. By replacing the origin of  $E$  by a fixed point of  $f$  if necessary, we may assume that  $f$  is an endomorphism of  $E$ .

Write  $g = \tau_Q \circ h$  where  $Q$  is a point on  $E$ ,  $\tau_Q$  is the map  $E \rightarrow E$  defined by translation by  $Q$ , and  $h$  is an endomorphism of  $E$ . Here  $\deg(h) = \deg(g) > 1$ , so that  $h - 1$  is nonconstant



and thus induces a surjective map  $E \rightarrow E$ . Let  $R$  be a point on  $E$  such that  $(h-1)(R) = Q$ . Then, for any positive integer  $n$ , we have

$$g^n = \tau_{Q+h(Q)+h^2(Q)+\dots+h^{n-1}(Q)} \circ h^n = \tau_{(h^n-1)(R)} \circ h^n. \quad (\text{II.3.1})$$

Thus, for any positive integer  $m$ , the condition  $f^m = g^n$  is equivalent to the conditions that  $f^m = h^n$  and  $(h^n - 1)(R) = O$ .

Pick the orbits of  $f$  and  $g$  that have infinite intersection, and let  $P$  be any point in the intersection; then the orbits  $O_f(P)$  and  $O_g(P)$  also have infinite intersection, so there are infinitely many pairs  $(m, n)$  of positive integers such that

$$f^m(P) = g^n(P) = (h^n - 1)(R) + h^n(P). \quad (\text{II.3.2})$$

Fix such a pair  $(m_0, n_0)$ , and let  $(m, n)$  be any other pair of positive integers that satisfy the above. Then

$$(f^{m_0} - h^{n_0})(P) = (h^{n_0} - 1)(R) \quad (\text{II.3.3})$$

$$(f^m - h^n)(P) = (h^n - 1)(R) \quad (\text{II.3.4})$$

Left-multiplying (II.3.3) by the dual isogeny  $(\bar{h}^{n_0} - 1)$  of  $(h^{n_0} - 1)$ , we get

$$(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0})(P) = \deg(h^{n_0} - 1)(R) \quad (\text{II.3.5})$$

Left-multiplying further by  $(h^n - 1)$ , we get

$$(h^n - 1)(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0})(P) = (h^n - 1) \deg(h^{n_0} - 1)(R) \quad (\text{II.3.6})$$

Note that  $\deg(h^{n_0} - 1)$  is an integer, so it is in the center of  $\text{End}(E)$ . Using (II.3.4), we get

$$\left( (h^n - 1)(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n) \deg(h^{n_0} - 1) \right) (P) = O. \quad (\text{II.3.7})$$

Since  $O_f(P)$  is infinite,  $P$  must be a point of infinite order (otherwise,  $rP = 0$  for some integer  $r > 0$ , so  $O_f(P)$  lies in the finite group  $E[r]$  of  $r$ -torion elements).

Hence the kernel of  $(h^n - 1)(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n) \deg(h^{n_0} - 1)$  contains all (infinitely many) multiples of  $P$ . Since the kernel of any nonzero endomorphism is a finite group, we must have

$$(h^n - 1)(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0}) - (f^m - h^n) \deg(h^{n_0} - 1) = 0 \quad (\text{II.3.8})$$

and recall that this holds for infinitely many pairs  $(m, n)$ .

Rewrite (II.3.8) as an equation in  $f^m$  and  $h^n$ :

$$h^n(u + d) - f^m d = u \tag{II.3.9}$$

where  $u = (\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0})$ ,  $d = \deg(h^{n_0} - 1)$ .

Now  $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$  is either  $\mathbb{Q}$  or an imaginary quadratic field or a positive definite quaternion algebra over  $\mathbb{Q}$ , all of which can be embedded into some positive definite quaternion algebra  $H$  over  $\mathbb{Q}$ . View the equation (II.3.9) in  $H$ .

If  $u \neq 0$ , then the equation  $h^n(u + d)u^{-1} - f^m d u^{-1} = 1$  has infinitely many solutions  $m, n > 0$ , a contradiction to Theorem II.1.1 with  $a = b = 1$ ,  $a' = (u + d)u^{-1}$ ,  $b' = -d u^{-1}$ ,  $\Gamma_1$  generated by  $h$ , and  $\Gamma_2$  generated by  $f$ . Hence  $u = 0$ , so that  $(\bar{h}^{n_0} - 1)(f^{m_0} - h^{n_0}) = 0$ .

But  $\deg \bar{h} = \deg h > 1$  implies  $\bar{h}^{n_0} - 1 \neq 0$ , so  $f^{m_0} = h^{n_0}$ .

Finally, equation (II.3.3) implies  $(h^{n_0} - 1)(R) = O$ , so  $g^{n_0} = h^{n_0} = f^{m_0}$  by (II.3.1). □

Let  $X$  be a simple abelian variety, and assume that Conjecture II.1.4 holds for  $A = \text{End}(X) \otimes \mathbb{Q}$  and  $\Gamma_1, \Gamma_2$  being cyclic semigroups. We claim that Corollary II.1.6 remains true if  $E$  is replaced by  $X$ . The proof is the same as above except for three places. First, we used the fact that  $h - 1$  is surjective because it is nonconstant. This is still true because the image of a morphism must be an abelian subvariety, but  $X$  is simple. Second, we used the elements  $\bar{h}^{n_0} - 1$ , but dual isogeny no longer exists in abelian varieties in general. However, we can fix an endomorphism  $\varphi$  such that  $\varphi \circ (h^{n_0} - 1) = \deg(h^{n_0} - 1)$ , and use  $\varphi$  in place of  $\bar{h}^{n_0} - 1$ . Third, we used the argument that if an endomorphism  $\psi$  of  $E$  vanishes at a point  $P$  of infinite order, then  $\psi = 0$ . This is also true for simple abelian variety  $X$ : the endomorphism  $\psi$  must vanish on the Zariski closure of the group generated by  $P$ , but it contains an abelian subvariety of  $X$  of positive dimension, which has to be the whole  $X$  because  $X$  is simple.

Given the referee's Proposition II.1.5, the ineffective part of the conditional result above holds unconditionally, and it gives an unconditional proof of Corollary II.1.7.

*Remark II.3.1.* When  $k$  has characteristic zero, the ineffective part of Corollary II.1.6 was proved via two different methods in [GN17, Theorem 1.4] and [Ode20, Theorem 1.2.3]. Our proof of Corollary II.1.6 extends the latter proof to arbitrary characteristic, and in fact the possibility of such an extension was the initial motivation for studying unit equations on quaternions in the present paper. We thank Michael Zieve for informing the author about [Ode20, Theorem 1.2.3] and suggesting this possibility.

*Remark II.3.2.* If  $f, g$  are endomorphisms of an elliptic curve  $E$  without translation, then Corollary II.1.6 becomes trivial. For a proof, set  $P \in E(k)$  be a point in the intersection

of orbits, and let  $n, m > 0$  be such that  $f^n(P) = g^m(P)$ . For any integer  $N$ , we have  $Nf^n(P) = Ng^m(P)$ , so that  $(f^n - g^m)(NP) = O$  because  $f, g$  are endomorphisms of  $E$ . But  $P$  is of infinite order (otherwise the forward orbit of  $P$  under  $f$  would be finite), so  $\ker(f^n - g^m)$  is an infinite group, and the only possibility is  $f^n - g^m = 0$ .

## II.4: Proof of Theorem II.1.2

Let  $\Delta$  be the set consisting of  $(f, g) \in \Gamma_1 \times \Gamma_2$  such that  $afa' + bgb' = 1$  and  $|1 - afa'| \neq |afa'|$ . Then the goal of Theorem II.1.2 is precisely to show that  $\Delta$  is a finite set.

By triangle inequality, every  $(f, g) \in \Delta$  satisfies

$$0 < \left| |afa'| - |bgb'| \right| \leq 1 \quad (\text{II.4.1})$$

We observe that since  $\Gamma_i$  ( $i = 1, 2$ ) is a semigroup generated by finitely many elements with norms greater than 1, there are only finitely many elements of  $\Gamma_i$  of bounded norm.

In the rest of the proof, we will prove the claim that  $\{|f| : (f, g) \in \Delta\}$  is bounded. Given the claim, the set  $\{f : (f, g) \in \Delta\}$  is finite by the observation above. Since  $f$  determines  $g$  by  $g = b^{-1}(1 - afa')b'^{-1}$ , there are only finitely many choices for  $g$  as well, and Theorem II.1.2 is proved.

For contradiction, we assume that there is a solution  $(f, g) \in \Delta$  with arbitrarily large  $|f|$ . Using simple calculus (specifically, Lagrange's mean value theorem), (II.4.1) implies

$$0 < \left| \log|afa'| - \log|bgb'| \right| \leq \frac{2}{|afa'|} \quad (\text{II.4.2})$$

for sufficiently large  $|f|$ .

Let the semigroup  $\log|\Gamma_1|$  be generated by  $x_1, \dots, x_t > 0$  and  $\log|\Gamma_2|$  by  $y_1, \dots, y_u > 0$ . Write  $\log|f| = m_1x_1 + \dots + m_tx_t$ ,  $\log|g| = n_1y_1 + \dots + n_uy_u$  for some nonnegative integers  $m_i, n_j$ . Let  $c = \log|aa'/bb'|$ . Then  $c, x_i, y_j$  are logarithms of real algebraic numbers, and (II.4.2) can be rewritten as

$$0 < \left| c + m_1x_1 + \dots + m_tx_t - n_1y_1 - \dots - n_uy_u \right| \leq \frac{2}{|a|e^{x_1m_1 + \dots + x_tm_t}} \quad (\text{II.4.3})$$

By Theorem II.2.1 (Baker's theorem), there are positive constants  $k, C$  such that

$$0 < \left| a_1c + m_1x_1 + \dots + m_tx_t - n_1y_1 - \dots - n_uy_u \right| \leq k \max\{|a_1|, |m_i|, |n_j|\}^{-C} \quad (\text{II.4.4})$$

has no integer solution  $(a_1, m_1, \dots, m_t, n_1, \dots, n_u)$ . In particular, for  $a_1 = 1$  and  $m_i, n_j > 0$ ,

the inequality

$$0 < |c + m_1x_1 + \cdots + m_tx_t - n_1y_1 - \cdots - n_uy_u| \leq kH^{-C} \text{ has no solution,} \quad (\text{II.4.5})$$

where  $H = \max\{1, m_1, \dots, m_t, n_1, \dots, n_u\}$ .

Our next goal is to bound the right-hand side of (II.4.3) by a function of  $H$ , in order to reach a contradiction with (II.4.5). Since  $x_i, y_j$  are positive, for  $|f|$  sufficiently large and satisfying (II.4.3), it is not hard to see that

$$C_1 \max\{m_i\} < \max\{n_j\} < C_2 \max\{m_i\} \quad (\text{II.4.6})$$

for some  $C_1, C_2 > 0$  that does not depend on  $m_i, n_j$ . For a proof, we note that

$$\min\{x_i\} \max\{m_i\} \leq m_1x_1 + \cdots + m_tx_t \leq t \max\{x_i\} \max\{m_i\} \quad (\text{II.4.7})$$

$$\min\{y_j\} \max\{n_j\} \leq n_1y_1 + \cdots + n_uy_u \leq u \max\{y_j\} \max\{n_j\} \quad (\text{II.4.8})$$

and (II.4.3) gives

$$\frac{1}{2}(n_1y_1 + \cdots + n_uy_u) < m_1x_1 + \cdots + m_tx_t < 2(n_1y_1 + \cdots + n_uy_u) \quad (\text{II.4.9})$$

for sufficiently large  $|f|$ . Hence  $\max\{m_i\}, \max\{n_j\}, \log|f|$  and  $\log|g|$  are all “comparable” to each other in the sense of (II.4.6)

It follows that

$$C_1 \max\{m_i\} < H \leq \max\{C_2, 1\} \max\{m_i\} =: C'_2 \max\{m_i\} \quad (\text{II.4.10})$$

where we denote  $C'_2 = \max\{C_2, 1\}$ .

Now (II.4.3) implies

$$0 < |c + m_1x_1 + \cdots + m_tx_t - n_1y_1 - \cdots - n_uy_u| \leq \frac{2}{|a|e^{\min\{x_i\} \max m_i}} \leq \frac{2}{|a|e^{\min\{x_i\} H/C'_2}} \quad (\text{II.4.11})$$

for sufficiently large  $|f|$  (or equivalently,  $H$ , by the “comparability” discussion above together with (II.4.10)).

Since the right-hand side decays exponentially in  $H$ , it will be less than  $kH^{-C}$  for large  $H$ , which contradicts the lack of solution of (II.4.5).  $\square$

## II.5: Proof of Theorem II.1.3

First, we observe that the equation  $|1 - afa'| = |afa'|$  can be rewritten as  $|a^{-1}a'^{-1} - f| = |0 - f|$ . Note that  $|\cdot|$  is the norm induced from the inner product on  $\mathbb{H}$  with  $\{1, i, j, k\}$  being an orthonormal basis. We denote the inner product by  $\langle \cdot, \cdot \rangle$ .

Denoting  $d = a^{-1}a'^{-1}$ , the equation above gives

$$\langle f, f \rangle = |f|^2 \tag{II.5.1}$$

$$= |d - f|^2 \tag{II.5.2}$$

$$= \langle d - f, d - f \rangle = |d|^2 - 2\langle d, f \rangle + |f|^2, \tag{II.5.3}$$

which simplifies to  $2\langle d, f \rangle = |d|^2$ .

Hence the equation is equivalent to that  $f$  lies in a hyperplane not passing through the origin, given by

$$\{x \in \mathbb{H} : \langle a^{-1}a'^{-1}, x \rangle = \frac{1}{2}|a^{-1}a'^{-1}|^2\}. \tag{II.5.4}$$

Given the observation above, Theorem II.1.3 follows from the following lemma:

**Lemma II.5.1.** *Let  $\Gamma$  be a commutative semigroup of  $\mathbb{H}^\times$  generated by finitely many algebraic elements of norms greater than 1, and  $H$  be a hyperplane of  $\mathbb{H}$  defined by*

$$H = \{x \in \mathbb{H} : \Theta(x) = 1\} \tag{II.5.5}$$

where  $\Theta : \mathbb{H} \rightarrow \mathbb{R}$  is a nonzero  $\mathbb{R}$ -linear functional that maps  $\mathbb{H}_a$  into  $\overline{\mathbb{Q}} \cap \mathbb{R}$ . Then  $\Gamma \cap H$  is finite. In fact, we have an effectively computable upper bound that depends only on  $H$  and  $\Gamma$  for norms of elements of  $\Gamma \cap H$ .

*Proof of lemma.* Since  $\Gamma$  is commutative, it lies in a subalgebra in  $\mathbb{H}$  that is isomorphic to  $\mathbb{C}$ . Passing to its restriction on this subalgebra, we may assume instead that  $\Gamma$  is a semigroup generated by  $g_1, \dots, g_s \in \overline{\mathbb{Q}}^\times \subseteq \mathbb{C}$  such that  $|g_j| > 1$ , and  $\Theta : \mathbb{C} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear functional (which could now be zero) that maps  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}} \cap \mathbb{R}$ . We need to show that  $\Theta(f) = 1$  has only finitely many solutions  $f \in \Gamma$ .

There is no question to ask if  $\Theta = 0$ . In the case  $\Theta \neq 0$ , we may assume  $\Theta$  is given by  $\langle v, \cdot \rangle$  for some nonzero vector  $v \in \overline{\mathbb{Q}}$ , where  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product on  $\mathbb{C}$  with  $\{1, i\}$  being an orthonormal basis. By rescaling, we may assume  $|v| = 1$ , but the equation  $\Theta(f) = 1$  will become

$$\langle v, f \rangle = M \tag{II.5.6}$$

for some real algebraic number  $M > 0$ .

Write  $g_j = r_j v_j$  with  $r_j > 1$  and  $v_j = e^{i\theta_j}$  on the unit circle, with  $0 \leq \theta_j < 2\pi$ . Also write  $v = e^{i\theta}$  with  $0 \leq \theta < 2\pi$ . For  $f = g_1^{n_1} \dots g_s^{n_s}$ , the equation (II.5.6) becomes

$$\langle v, e^{i(n_1\theta_1 + \dots + n_s\theta_s)} \rangle = Mr_1^{-n_1} \dots r_s^{-n_s} \quad (\text{II.5.7})$$

The left-hand side involves the inner product of two unit vectors, so its value is  $\cos((n_1\theta_1 + \dots + n_s\theta_s) - \theta)$ . When  $n_i$  are sufficiently large, the right-hand side of II.5.7 is small. But  $|\cos((n_1\theta_1 + \dots + n_s\theta_s) - \theta)|$  is approximately the closest distance from  $(n_1\theta_1 + \dots + n_s\theta_s) - \theta$  to  $(m + 1/2)\pi$  for integer  $m$ . If (II.5.7) is satisfied by infinitely many  $(n_j)$ 's, then for sufficiently large solutions  $(n_j)$ , we have

$$0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \dots + n_s\theta_s) \right| < 2Mr_1^{-n_1} \dots r_s^{-n_s} \quad (\text{II.5.8})$$

for some  $m \in \mathbb{Z}$ .

By assumption,  $v, v_j$  are algebraic numbers, so  $\lambda := i(\frac{1}{2}\pi + \theta)$ ,  $\mu = i\pi$  and  $\lambda_j = i\theta_j$  are logarithms of algebraic numbers. By Theorem II.2.1, there are constants  $k, C > 0$  such that the inequality

$$0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \dots + n_s\theta_s) \right| < kB^{-C} \text{ has no solution} \quad (\text{II.5.9})$$

for  $m, n_j \in \mathbb{Z}, n_j \geq 0$ , where

$$B = \max\{1, |m|, n_j\} \quad (\text{II.5.10})$$

But for solutions of (II.5.8) with  $n_j$  large,  $m\pi$  must be close to  $n_1\theta_1 + \dots + n_s\theta_s - (\frac{1}{2}\pi + \theta)$ . Noting that

$$n_1\theta_1 + \dots + n_s\theta_s \leq s \max\{\theta_j\} \max\{n_j\}, \quad (\text{II.5.11})$$

we have

$$|m| \leq C' \max\{n_j\} \quad (\text{II.5.12})$$

for some constant  $C'$ , and thus

$$\max\{n_j\} \leq B = \max\{n_j, |m|\} \leq \max\{1, C'\} \max\{n_j\} \quad (\text{II.5.13})$$

It follows from (II.5.9) that for some constant  $k' > 0$ ,

$$0 < \left| \left( \frac{\pi}{2} + \theta \right) + m\pi - (n_1\theta_1 + \dots + n_s\theta_s) \right| < k' \max\{n_j\}^{-C} \text{ has no solution} \quad (\text{II.5.14})$$

for  $m, n_j \in \mathbb{Z}, n_j \geq 0$ . But for  $(n_j)$  large,  $2Mr_1^{-n_1} \dots r_s^{-n_s} < k' \max\{n_j\}^{-C}$ , yielding a

contradiction with (II.5.8). □

## II.6: Future Work

We were able to arrive at the main theorem using the archimedean norm only. If we can furthermore use some version of  $p$ -adic norm on the division algebra  $A$ , we can vastly improve the result by applying K. Yu's theorem about  $p$ -adic logarithms in [Yu07]. One possible proposal for a  $p$ -adic norm is to use the reduced norm of a division algebra over  $\mathbb{Q}_p$ , which only works if  $A \otimes \mathbb{Q}_p$  is still a division algebra. Unfortunately, for each given  $A$ , this only holds for finitely many  $p$ .

Theorem II.1.2 is potentially useful for more cases than in Theorem II.1.1. For example, one can explore the analogue of Theorem II.1.3 in the case where  $\Gamma$  has two or more non-commutative generators, and then apply Theorem II.1.2. Even if  $\Gamma$  is replaced by its subset  $\{f_1^{n_1} f_2^{n_2} : n_1, n_2 \geq 0\}$ , where  $f_1, f_2$  are noncommutative generators with norms greater than 1, the analogue of Theorem II.1.3 remains open.

The following example shows that we should only consider Conjecture II.1.4 where  $A$  is a division algebra.

**Example II.6.1.** Take  $A = M_2(\mathbb{Q})$ , the algebra of  $2 \times 2$  matrices over  $\mathbb{Q}$ . Then the multiplicative semigroup generated by  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  is

$$\Gamma := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z}, n \geq 0 \right\}. \tag{II.6.1}$$

The equation  $2f - g = 1_A$  has infinitely many solutions  $f, g \in \Gamma$ , namely all  $(f, g)$  with  $f \in \Gamma$  and  $g = f^2$ .

## II.7: Appendix

This section contains the proofs of Proposition II.1.5 and Proposition II.7.2, both sketched by the referee. We start with an observation that will be used in both proofs.

**Lemma II.7.1.** *Let  $A$  be a finite-dimensional division algebra over a field  $k$ , and let  $K$  be the algebraic closure of  $k$ . Then there is an embedding of  $k$ -algebras from  $A$  to the matrix algebra  $M_n(K)$  for some integer  $n > 0$ .*

*Proof.* Let  $L$  be the center of  $A$ . Then  $L$  is a finite extension of  $k$  and we can embed  $A$  into

$A \otimes_L K$ , which is isomorphic to the matrix algebra  $M_n(K)$  for some integer  $n$  by a standard fact about central simple algebras.  $\square$

*Proof of Proposition II.1.5.* Let  $K$  be the algebraic closure of  $k$  and fix an embedding  $A \hookrightarrow M_n(K)$  as in Lemma II.7.1. Note that nonzero elements of  $A$  are sent to invertible matrices in  $M_n(K)$ . From now on, we shall consider the unit equation in  $M_n(K)$ .

To set up a proof by contradiction, we assume that

$$a_1 f_1 b_1 + \cdots + a_m f_m b_m = 1, (f_1, \dots, f_m) \in \Gamma_1 \times \cdots \times \Gamma_m \quad (\text{II.7.1})$$

is a shortest equation (i.e., with minimal  $m$ ) in the setting of Proposition II.1.5 that has infinitely many degenerate solutions. We claim:

There cannot exist an infinite family of degenerate solutions  $\{(f_1^\alpha, \dots, f_m^\alpha)\}$  indexed by  $\alpha$  in an infinite set, such that  $f_1^\alpha$  are the same for all  $\alpha$ . (II.7.2)

Otherwise, call  $f_1^\alpha = f_1$ , and let  $u = a_1 f_1 b_1$ , which is not 1 because the solution is nondegenerate. Then  $1 - u$  is a unit in  $A$  because  $A$  is a division algebra, and set  $b'_i = b_i(1 - u)^{-1}$ . The following equation

$$a_2 f_2 b'_2 + \cdots + a_m f_m b'_m = 1 \quad (\text{II.7.3})$$

has infinitely many nondegenerate solutions  $(f_2, \dots, f_m) = (f_2^\alpha, \dots, f_m^\alpha)$ , contradicting the minimality of  $m$ .

Now note that every element  $\gamma$  of  $\Gamma_i$  is diagonalizable in  $M_n(K)$ . Indeed, since  $\gamma \in R$  and  $R$  is finite-dimensional over  $k$ , we see that  $\gamma$  satisfies some minimal polynomial  $p(\gamma) = 0$  where  $p(x) \in k[x]$  is monic. To show that  $\gamma$  is diagonalizable, it suffices to show that  $p(x)$  has no repeated root in  $K$ . Assume the contrary, then there is a proper divisor  $p_0(x) \in k[x]$  of  $p(x)$  such that  $p(x)$  divides  $p_0(x)^2$ . Thus  $p_0(\gamma)^2 = 0$ , so that  $p_0(\gamma) = 0$  because  $R$  is a division algebra. This is a contradiction to the minimality of  $p(x)$ .

Since  $\Gamma_i$  is abelian and finitely generated, and every element of  $\Gamma_i$  is diagonalizable, there is a simultaneous diagonalization of  $\Gamma_i$  by some  $s_i \in GL_n(K)$ , i.e.,  $s_i \Gamma_i s_i^{-1}$  only consists of diagonal matrices in  $M_n(K)$ . So we may replace  $a_i$  by  $a_i s_i^{-1}$ ,  $b_i$  by  $s_i b_i$ , and  $\Gamma_i$  by  $s_i \Gamma_i s_i^{-1}$  and assume that each  $\Gamma_i$  only consists of diagonal matrices in  $M_n(K)$  (though  $\Gamma_i, a_i, b_i$  are no longer inside  $R$ , but it will not matter).

Now consider a solution  $(f_i)$  of  $1 = a_1 f_1 b_1 + \cdots + a_m f_m b_m$ , where  $f_i = \text{diag}(x_{i1}, \dots, x_{in})$ .



Looking at the  $(1, 1)$ -entry, we obtain an equation of the form

$$1 = \sum_{1 \leq i, j \leq n} p_{ij} x_{ij} \quad (\text{II.7.4})$$

for some fixed  $p_{ij} \in K$ . (To see it, one merely needs to notice that every entry of  $a_1 f_1 b_1 + \dots + a_m f_m b_m$  is a linear combination of entries of  $f_i$ .)

Let  $S$  be the set consisting of all  $(i, j)$  such that  $p_{ij} \neq 0$ . We may assume that  $S$  is nonempty, otherwise (II.7.4) has no solution and there is nothing to prove. We claim that

$$\begin{aligned} &\text{There are finite sets } X_{ij} \text{ for } (i, j) \in S, \text{ such that whenever } (x_{ij}) \text{ is} \\ &\text{a solution of (II.7.4), there exists } (i_0, j_0) \in S \text{ such that } x_{i_0 j_0} \in X_{i_0 j_0}. \end{aligned} \quad (\text{II.7.5})$$

To prove the claim, notice that there is a finitely generated subgroup of  $K^\times$  that contains all  $x_{ij}$  because  $\Gamma_i$  is finitely generated. By the  $S$ -unit theorem in several variables [EG15, Theorem 6.1.3], for every nonempty subset  $T$  of  $S$ , the equation

$$1 = \sum_{(i,j) \in T} p_{ij} x_{ij} \quad (\text{II.7.6})$$

has only finitely many nondegenerate solutions. Let  $X_{ij}$  be the set consisting of all  $x_{ij}$  that appears in a nondegenerate solution of (II.7.6) for some  $T$ . Now for every solution  $(x_{ij})$  of (II.7.4), there is a minimal nonempty subset  $T_0$  of  $S$  such that  $1 = \sum_{(i,j) \in T_0} p_{ij} x_{ij}$ . Pick  $(i_0, j_0) \in T_0$ , then by construction,  $x_{i_0 j_0} \in X_{i_0 j_0}$ , as required.

Now applying the pigeonhole principle to the infinitely many solutions of (II.7.1), we see that there exists  $(i_0, j_0) \in S$  and  $x \in X_{i_0 j_0}$  such that there are infinitely many solutions  $(f_i) = (\text{diag}(x_{ij}))$  with  $x_{i_0 j_0} = x$ . Without loss of generality, assume  $(i_0, j_0) = (1, 1)$ . We claim that all those solutions  $(f_i)$  have the same  $f_1$ . This contradicts (II.7.2) and finishes the proof of Proposition II.1.5.

It remains to prove the claim. Let  $(f_i) = \text{diag}(x_{i1}, \dots, x_{in})$  and  $(f'_i) = \text{diag}(x'_{i1}, \dots, x'_{in})$  be two solutions of (II.7.1) such that  $x_{11} = x'_{11} = x$ . Then  $g := f_1 (f'_1)^{-1}$  is a diagonal matrix with  $(1, 1)$ -entry being 1. In particular,  $g - 1$  is not invertible in  $M_n(K)$ . But by construction,  $h := s_1^{-1} (g - 1) s_1$  is in  $A$ , so  $h$  is either zero or a unit. Since  $g - 1$  is not invertible, neither is  $h$ . Therefore  $h = 0$ , so that  $g = 1$ , which gives  $f_1 = f'_1$ .  $\square$

We next prove a statement that implies a generalization of Proposition II.1.5 where all  $\Gamma_i$  are semigroups not containing free semigroups of rank two. Recall that a group  $G$  is *virtually*  $P$  (where  $P$  is a property) if  $G$  has a finite-index subgroup that is  $P$ .

**Proposition II.7.2.** *Let  $A$  be a division algebra over a field  $k$ , and  $\Gamma$  be a finitely generated semigroup of  $A^\times$ . If  $\Gamma$  does not contain a free semigroup of rank two, then the group  $G$  generated by  $\Gamma$  in  $A^\times$  is virtually abelian.*

*Proof.* First, we will show that  $G$  is virtually nilpotent using Theorem 1 of [OS95]. Embed  $A$  into  $M_n(K)$  as in Lemma II.7.1, where  $K$  is the algebraic closure of  $k$ . Note that  $\Gamma$  is in  $GL_n(K)$ .

Since  $\Gamma$  is finitely generated, there is a finitely generated subfield  $L \subseteq K$  such that  $\Gamma \subseteq GL_n(L)$  (for example, let  $L$  be the field generated by matrix entries of generators of  $\Gamma$  over the prime field  $\mathbb{Q}$  or  $\mathbb{F}_p$  of  $k$ ). Now, Theorem 1 of [OS95] implies that  $G$  is virtually nilpotent.

Let  $N$  be a finite-index nilpotent subgroup of  $G$ . By the following lemma,  $N$  is virtually abelian, and the proof is complete.  $\square$

*Warning.* Here  $L$  may not contain  $k$ , but it does not matter for the purpose of this proof.

**Lemma II.7.3.** *If  $A$  is a division algebra over a field  $k$ , and  $N \subseteq A^\times$  is a solvable subgroup, then  $N$  is virtually abelian.*

*Proof.* Again, we embed  $A$  into  $M_n(K)$  as in Lemma II.7.1, where  $K$  is the algebraic closure of  $k$ . Then  $N$  is a subgroup of  $GL_n(K)$ . The Zariski closure  $\overline{N}$  of  $N$  in  $GL_n(K)$  is a  $K$ -algebraic group that is still solvable, and so is its identity component  $\overline{N}_0$ . Let  $N_0 := \overline{N}_0 \cap N$ . Since  $\overline{N}_0$  has finite index in  $\overline{N}$ , so does  $N_0$  in  $N$ .

We claim that  $N_0$  is abelian. By the Lie–Kolchin triangularization theorem [Spr98, Theorem 6.3.1], there is  $s \in GL_n(K)$  such that  $s\overline{N}_0s^{-1}$  consists of upper triangular matrices, so  $sN_0s^{-1}$  does as well. We observe that if  $a$  and  $b$  are invertible upper triangular matrices, then the commutator  $[a, b] := aba^{-1}b^{-1}$  is in  $U$ , the group of upper triangular matrices with diagonal entries all 1. It follows that  $s[N_0, N_0]s^{-1} \subseteq U$ .

It remains to show that  $[N_0, N_0]$  is the trivial group. Take  $x \in [N_0, N_0]$ , and note that  $x \in A$ , so  $x - 1 \in A$  is either zero or invertible. But  $s(x - 1)s^{-1} = sx s^{-1} - 1$  is an upper triangular matrix in  $M_n(K)$  with diagonal entries all 0, so  $x - 1$  cannot be invertible. It follows that  $x = 1$  and  $N_0$  is abelian.  $\square$

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# CHAPTER III

## Mutually Annihilating Matrices, and a Cohen–Lenstra Series for the Nodal Singularity

The content of this chapter is available at [\[Hua21\]](#).

### Abstract

We give a generating function for the number of pairs of  $n \times n$  matrices  $(A, B)$  over a finite field that are mutually annihilating, namely,  $AB = BA = 0$ . This generating function can be viewed as a singular analogue of a series considered by Cohen and Lenstra. We show that this generating function has a factorization that allows it to be meromorphically extended to the entire complex plane. We also use it to count pairs of mutually annihilating nilpotent matrices. This work is essentially a study of the motivic aspects about the variety of modules over  $\mathbb{C}[u, v]/(uv)$  as well as the moduli stack of coherent sheaves over an algebraic curve with nodal singularities.

### III.1: Introduction

#### III.1.1: History and Motivation

Let  $R$  be a commutative ring with only finite quotient fields. We define the *Cohen–Lenstra zeta function* of  $R$  as

$$\widehat{\zeta}_R(s) := \sum_M \frac{1}{|\mathrm{Aut} M|} |M|^{-s}, \quad (\text{III.1.1})$$

where  $M$  ranges over all isomorphism classes of finite(-cardinality)  $R$ -modules.

When  $R$  is a Dedekind domain, this function agrees with a function defined in the important work of Cohen and Lenstra [\[CL84\]](#) about the statistics of finite  $R$ -modules, motivated by the distribution of class groups of imaginary quadratic fields. They defined  $\widehat{\zeta}_R(s)$  and proved a simple formula for it in [\[CL84, p. 39\]](#), which was crucial in their work.

If  $R$  contains a finite field  $\mathbb{F}_q$  with  $q$  elements, we define the *Cohen–Lenstra series* of  $R$  over  $\mathbb{F}_q$  as

$$\widehat{Z}_{R/\mathbb{F}_q}(x) := \sum_M \frac{1}{|\mathrm{Aut} M|} x^{\dim_{\mathbb{F}_q} M}, \quad (\text{III.1.2})$$

where  $M$  ranges over all isomorphism classes of finite  $R$ -modules. Clearly  $\widehat{\zeta}_R(s) = \widehat{Z}_{R/\mathbb{F}_q}(q^{-s})$ . When the ground field is clear from the context, we may simply denote  $\widehat{Z}_{R/\mathbb{F}_q}(x)$  by  $\widehat{Z}_R(x)$ .

Existing work in various areas of mathematics can be put in the context of the Cohen–Lenstra series:

- When  $R = \mathbb{F}_q$ , the series  $\widehat{Z}_{\mathbb{F}_q}(x)$  is the subject of Rogers–Ramanujan identities [And98, p. 104], which state that  $\widehat{Z}_{\mathbb{F}_q}(1)$  and  $\widehat{Z}_{\mathbb{F}_q}(q^{-1})$  each equals to an infinite product:

$$\widehat{Z}_{\mathbb{F}_q}(1) = \frac{1}{(q^{-1}; q^{-5})_{\infty} (q^{-4}; q^{-5})_{\infty}}; \quad (\text{III.1.3})$$

$$\widehat{Z}_{\mathbb{F}_q}(q^{-1}) = \frac{1}{(q^{-2}; q^{-5})_{\infty} (q^{-3}; q^{-5})_{\infty}}. \quad (\text{III.1.4})$$

- When  $R$  is the power series ring  $\mathbb{F}_q[[t]]$ , giving a formula of  $\widehat{Z}_R(x)$  is equivalent to finding the number of nilpotent matrices over  $\mathbb{F}_q$ , which was given by Fine and Herstein [FH58].
- When  $R = \mathbb{F}_q[u, v]$ , the series  $\widehat{Z}_R(x)$  is the generating function evaluated by Feit and Fine [FF60] to enumerate pairs of commuting matrices.
- When  $R = \mathbb{F}_q[[u, v]]$ , the series  $\widehat{Z}_R(x)$  is the generating function evaluated by Fulman and Guralnick [FG18] to enumerate pairs of commuting nilpotent matrices.

For any variety<sup>1</sup>  $X$  over  $\mathbb{F}_q$ , we define the *Cohen–Lenstra series* of  $X$  over  $\mathbb{F}_q$  as

$$\widehat{Z}_{X/\mathbb{F}_q}(x) := \sum_M \frac{1}{|\mathrm{Aut} M|} x^{\dim_{\mathbb{F}_q} H^0(X; M)}, \quad (\text{III.1.5})$$

where  $M$  ranges over all isomorphism classes of finite-length coherent sheaves over  $X$ , and  $H^0(X; M)$  denotes the space of global sections of  $M$ . This generalizes the Cohen–Lenstra series over a ring, since  $\widehat{Z}_R(x) = \widehat{Z}_{\mathrm{Spec} R}(x)$ . If  $p$  is a closed point of  $X$ , we define the *local Cohen–Lenstra series* of  $X$  over  $p$  as

$$\widehat{Z}_{X,p}(x) := \widehat{Z}_{\mathcal{O}_{X,p}}(x) = \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x), \quad (\text{III.1.6})$$

where  $\mathcal{O}_{X,p}$  is the local ring of  $X$  at  $p$  and  $\widehat{\mathcal{O}}_{X,p}$  is its completion.

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<sup>1</sup>separated scheme of finite type

It is implicit in the work of Bryan and Morrison [BM15] (and references therein) that  $\widehat{Z}_X(x)$  and  $\widehat{Z}_{X,p}(x)$  are known if  $X$  is a smooth curve or a smooth surface (Proposition III.4.5). The Cohen–Lenstra series satisfies an important “Euler product” property (Proposition III.4.2): for any variety  $X$ , we have

$$\widehat{Z}_X(x) = \prod_{p \in \text{cl}(X)} \widehat{Z}_{X,p}(x), \quad (\text{III.1.7})$$

where  $\text{cl}(X)$  denotes the set of closed points of  $X$ . In light of this property, the study of  $\widehat{Z}_X(x)$  is equivalent to the study of the local factors  $\widehat{Z}_{X,p}(x)$ . When  $X$  is a reduced curve or surface, the only unknown factors are those at singular points. We can view the local Cohen–Lenstra series as an invariant attached to the classification of singularities up to analytic isomorphism, so it is natural to wonder what  $\widehat{Z}_{X,p}(x)$  reveals about the geometry of  $X$  at  $p$ .

### III.1.2: Main results

The goal of this paper is to determine the properties of  $\widehat{Z}_{X,p}(x)$  where  $X$  is a singular curve over  $\mathbb{F}_q$  and  $p$  is a *nodal singularity*, namely, a singularity whose completed local ring is isomorphic to  $\mathbb{F}_q[[u, v]]/(uv)$ . This is the first result about the local Cohen–Lenstra series of a singularity. We use the  $q$ -Pochhammer symbol

$$(a; q)_n := (1 - a)(1 - qa) \dots (1 - q^{n-1}a), \quad (\text{III.1.8})$$

$$(a; q)_\infty := (1 - a)(1 - qa)(1 - q^2a) \dots \quad (\text{III.1.9})$$

**Theorem III.1.1.** *Fix a prime power  $q > 1$  and let  $R_q = \mathbb{F}_q[[u, v]]/(uv)$ . Then*

- (a) *The power series  $\widehat{Z}_{R_q}(x)$  in  $x$  has a meromorphic continuation to all of  $\mathbb{C}$ .*
- (b) *The poles of the meromorphic continuation of  $\widehat{Z}_{R_q}(x)$  are precisely double poles at  $x = q^i$ ,  $i = 1, 2, \dots$ . Moreover, the power series  $\widehat{Z}_{R_q}(x)$  admits a factorization*

$$\widehat{Z}_{R_q}(x) = \frac{1}{(xq^{-1}; q^{-1})_\infty^2} H_q(x) \quad (\text{III.1.10})$$

where

$$H_q(x) := \sum_{k=0}^{\infty} \frac{q^{-k^2} x^{2k}}{(q^{-1}; q^{-1})_k} (xq^{-k-1}; q^{-1})_\infty. \quad (\text{III.1.11})$$

We point out that  $H_q(x)$  is an entire power series in  $x$  whenever  $|q| > 1$ .

$$(c) \widehat{Z}_{R_q}(1) = \frac{1}{(q^{-1}; q^{-1})_\infty^2}, \text{ and } \widehat{Z}_{R_q}(-1) = \frac{1}{(-q^{-2}; q^{-2})_\infty}.$$

Since  $R_q = \mathbb{F}_q[[u, v]]/(uv)$  is the completed local ring of a curve at a nodal singularity, the series  $\widehat{Z}_{R_q}(x)$  in Theorem III.1.1 is precisely the local Cohen–Lenstra series of a nodal singularity. Theorem III.1.1(b) implies that the series  $\widehat{Z}_{R_q}(x)$  has a radius of convergence equal to  $q$ ; the content of Theorem III.1.1(a)(b) lies in the description of  $\widehat{Z}_{R_q}(x)$  outside the domain of convergence. The specialization of  $\widehat{Z}_{R_q}(x)$  to  $x = \pm 1$  is especially important because of the statistical interpretations below. The value of  $\widehat{Z}_{R_q}(1)$  is the weighted count of finite-cardinality  $R$ -modules up to isomorphism, each weighted inversely by the size of the automorphism group. The numbers  $(\widehat{Z}_{R_q}(1) \pm \widehat{Z}_{R_q}(-1))/2$  give the weighted counts of even- and odd-dimensional  $R_q$ -modules up to isomorphism, respectively. Theorem III.1.1(c) thus gives a clean formula for all of the aforementioned weighted counts, and the cleanness is surprising given that the classification of  $R_q$ -modules (see [BGS09] and its references) is much more complicated than the classically well-known classification of  $\mathbb{F}_q[[t]]$ -modules. For comparison, let  $S_q = \mathbb{F}_q[[t]]$ , the completed local ring of a curve at a smooth  $\mathbb{F}_q$ -point. We have

$$\widehat{Z}_{R_q}(x) = \frac{1}{(xq^{-1}; q^{-1})_\infty^2} H_q(x); \tag{III.1.12}$$

$$\widehat{Z}_{S_q}(x) = \frac{1}{(xq^{-1}; q^{-1})_\infty}. \tag{III.1.13}$$

The series  $\widehat{Z}_{R_q}(x)$  has a complicated factor  $H_q(x)$  not present in  $\widehat{Z}_{S_q}(x)$ ; we may interpret  $H_q(x)$  as a factor accounting for the nodal singularity. However, specializing to  $x = \pm 1$ , the values of  $\widehat{Z}_{R_q}(x)$  are no longer “much more complicated” than  $\widehat{Z}_{S_q}(x)$ , as

$$\widehat{Z}_{R_q}(1) = \frac{1}{(q^{-1}; q^{-1})_\infty^2}; \tag{III.1.14}$$

$$\widehat{Z}_{R_q}(-1) = \frac{1}{(-q^{-2}; q^{-2})_\infty}; \tag{III.1.15}$$

$$\widehat{Z}_{S_q}(\pm 1) = \frac{1}{(\pm q^{-1}; q^{-1})_\infty}. \tag{III.1.16}$$

Theorem III.1.1 also implies that if  $X$  is a reduced curve with only nodal singularities, and  $\widetilde{X}$  is its resolution of singularity, then  $\widehat{Z}_X(x)/\widehat{Z}_{\widetilde{X}}(x)$  is entire. This can be interpreted as that the Cohen–Lenstra series of a nodal singular curve, while being mysterious, is “not too far” from its smooth version.

Our proof of Theorem III.1.1 depends on the following combinatorial identity.

**Theorem III.1.2.** *As power series in  $x$ , we have*

$$\sum_{n=0}^{\infty} \frac{|\{(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \frac{1}{(x; q^{-1})_{\infty}^2} H_q(x), \quad (\text{III.1.17})$$

where  $H_q(x)$  is defined in (III.1.11).

The content of Theorem III.1.2 lies not only in the enumeration of pairs of mutually annihilating matrices, but also in the unusual factorization identity (III.1.17). We point out that the left-hand side of (III.1.17) is precisely  $\widehat{Z}_{\mathbb{F}_q[u,v]/(uv)}(x)$  (note the single bracket). We also point out that obtaining the specific expression of  $H_q(x)$  in (III.1.11) is the key to prove Theorem III.1.1. It is currently unknown if any part of Theorem III.1.1 has a geometric proof.

We point out that (III.1.10) gives a formula that counts pairs of mutually annihilating *nilpotent* matrices. In specific,

$$\sum_{n=0}^{\infty} \frac{|\{(A, B) \in \text{Nilp}_n(\mathbb{F}_q) \times \text{Nilp}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \frac{1}{(xq^{-1}; q^{-1})_{\infty}^2} H_q(x), \quad (\text{III.1.18})$$

where  $\text{Nilp}_n(\mathbb{F}_q)$  denotes the set of  $n$  by  $n$  nilpotent matrices over  $\mathbb{F}_q$ , and  $H_q(x)$  is defined in (III.1.11). Pairs of mutually annihilating nilpotent matrices are much harder to count than pairs of mutually annihilating matrices, so deriving (III.1.10) from Theorem III.1.2 can be viewed as an application of the Cohen–Lenstra series and its Euler product property.

As an attempt to extend Theorem III.1.1 to other curve singularities, we formulate the following questions:

**Question III.1.3.** Let  $p$  be an  $\mathbb{F}_q$ -point of a reduced curve  $X$ .

- (a) Is it always true that  $\widehat{Z}_{X,p}(x)$  has a meromorphic continuation to all of  $\mathbb{C}$ ?
- (b) If the answer to (a) is yes, is it true that the poles of  $\widehat{Z}_{X,p}(x)$  are given by the factor  $(xq^{-1}; q^{-1})_{\infty}^{-r(p)}$ , where  $r(p)$  is some numeric invariant attached to the pair  $(X, p)$ ?
- (c) Do the special values  $\widehat{Z}_{X,p}(\pm 1)$  read the geometry of  $X$  at  $p$  in a meaningful way?

A possibility that is compatible to all the known cases (i.e., smooth point and nodal singularity) is that  $r(p)$  is the branching number of  $X$  at  $p$ . This specific choice of  $r(p)$  would imply an elegant global statement that whenever  $\widetilde{X}$  is a resolution of singularity of a reduced curve  $X$ , the quotient power series  $\widehat{Z}_X(x)/\widehat{Z}_{\widetilde{X}}(x)$  is entire. An analogous quotient about the Hilbert schemes of points on a curve with planar singularities was considered in



[GS14, p. 2259], where it was shown to be a polynomial as opposed to a rational function with a nontrivial denominator.

### III.1.3: Related work

The coefficients of the Cohen–Lenstra series encode the point count of a wide family of varieties that arise as variants of the commuting variety. The commuting variety is the variety of pairs of commuting matrices, whose geometry was studied by Motzkin and Tausky [MT55] and Gerstenhaber [Ger61]. Generalizations and variants of the commuting variety have been introduced and their geometry has been studied in both characteristic zero and positive characteristic; see [Bar01, CL19, CW20, CBS02, Ric79].

One of the variants of the commuting variety is the variety of modules [CBS02]. Our Theorem III.1.1 and III.1.2, in particular, give the point count of the varieties of modules over  $\mathbb{F}_q[[u, v]]/(uv)$  and  $\mathbb{F}_q[u, v]/(uv)$ , respectively. Their point count can be viewed as a statistical information (in the sense of Cohen–Lenstra [CL84]) about the classification of finite-dimensional modules over  $\mathbb{F}_q[u, v]/(uv)$  up to isomorphism. See [BGS09, NRSB75, Sch04] for studies of the structure of these varieties and the aforementioned classification problem. See also [MR18] for similar work about  $\mathbb{F}_q[u, v]$ .

For an  $\mathbb{F}_q$ -variety  $X$ , the coefficients of  $\widehat{Z}_X(x)$  are precisely the point count of the motivic class of the moduli stack of coherent sheaves over  $X$ . Bryan and Morrison [BM15] reproved and refined the result of Feit and Fine about  $\widehat{Z}_{\mathbb{F}_q[u, v]}$  from this motivic point of view. Much more is known motivically about a related moduli space, namely, the Hilbert scheme of points on  $X$ , not only when  $X$  is a smooth surface [Che94, G90, G01], but also when  $X$  is a curve with planar singularities [GS14, MY14], and even for some singular surfaces [GNS17]. One of the motivations of our work is to find analogy between known motivic statements about the Hilbert scheme and about the stack of coherent sheaves. Despite some known geometric connections between the Hilbert scheme and the stack of coherent sheaves [HJ18, Nak99], it is unknown if they are motivically related in some way.

### III.1.4: Organization of the paper

In Section III.2, we give some preliminaries about partitions and  $q$ -series that will be used in the proof of Theorem III.1.2, given in Section III.3. In Section III.4, we give a self-contained introduction to known properties of the Cohen–Lenstra series, part of which will be used to prove Theorem III.1.1. In Section III.5, we make some elementary observations about the series  $H_q(x)$  appearing in the main theorems, and use them to finish the proof of Theorem III.1.1. In Section III.6, we discuss a possible connection between  $H_q(x)$  and the partial

theta function.

### III.2: Preliminaries

A *partition*  $\lambda$  is a finite nonincreasing sequence of positive integers  $(\lambda_1, \dots, \lambda_\ell)$ , each of which is called a *part* of  $\lambda$ . The *length* of  $\lambda$  is the number of parts in  $\lambda$ , denoted  $\ell(\lambda)$ . The *size* of  $\lambda$  is  $|\lambda| := \sum_i \lambda_i$ . We denote by  $a_i(\lambda)$  the number of parts of  $\lambda$  of size  $i$ , so we can write down a partition as

$$\lambda = a_1(\lambda) \cdot [1] + a_2(\lambda) \cdot [2] + \dots \quad (\text{III.2.1})$$

The *Young diagram* of  $\lambda$  follows the convention such that it has  $\lambda_1$  boxes in the top row, and  $\ell(\lambda)$  boxes in the leftmost column. We will often refer to a partition by its Young diagram. The *conjugate* (or *transpose*) partition of  $\lambda$  is the partition, denoted  $\lambda'$ , whose Young diagram is the transpose of the Young diagram of  $\lambda$ .

The (first) *Durfee square* of  $\lambda$  is the largest square that fits the top-left corner of its Young diagram. For  $i \geq 1$ , the  $(i+1)$ -st Durfee square is the Durfee square of the part of  $\lambda$  below the  $i$ -th Durfee square. We denote the sidelength of the  $i$ -th Durfee square by  $\sigma_i(\lambda)$ , and define the *Durfee partition* as

$$\text{Durf}(\lambda) := (\sigma_1(\lambda), \sigma_2(\lambda), \dots). \quad (\text{III.2.2})$$

Recall the *q-Pochhammer symbol*

$$(a; q)_n = (1-a)(1-qa) \dots (1-q^{n-1}a); \quad (\text{III.2.3})$$

$$(a; q)_\infty = (1-a)(1-qa)(1-q^2a) \dots \quad (\text{III.2.4})$$

The *q-binomial coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \quad (\text{III.2.5})$$

### III.3: Proof of Theorem III.1.2

We define  $\widehat{Z}(x) := \sum_{n=0}^{\infty} \frac{|\{A, B \in \text{Mat}_n(\mathbb{F}_q) : AB = BA = 0\}|}{|\text{GL}_n(\mathbb{F}_q)|} x^n$  and compute it in two steps.

### III.3.1: Counting pairs of mutually annihilating matrices

Fix  $n$ . We count the number of pairs  $(A, B)$  of  $n \times n$  matrices such that  $AB = BA = 0$ .

First, we fix  $A$  and let  $0 \leq k \leq n$  be the nullity of  $A$  (so the rank of  $A$  is  $n - k$ ). Let  $\text{im } A = V$ ,  $\ker A = W$ , then  $\dim V = n - k$ ,  $\dim W = k$ . We have

$$AB = 0 \iff A(\text{im } B) = 0 \iff \text{im } B \subseteq W; \quad (\text{III.3.1})$$

$$BA = 0 \iff B(\text{im } A) = 0 \iff \ker B \supseteq V. \quad (\text{III.3.2})$$

Hence, choosing  $B$  that mutually annihilates  $A$  is equivalent to picking a linear map from  $\mathbb{F}_q^n/V \rightarrow W$ . Since  $\dim \mathbb{F}_q^n/V = \dim W = k$ , there are  $q^{k^2}$  choices of  $B$ . Notice that this number depends only on the rank of  $A$ .

It is a standard fact that the number of  $n \times n$   $\mathbb{F}_q$ -matrices of nullity  $k$  is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q (q^n - 1)(q^n - q) \dots (q^n - q^{n-k-1}). \quad (\text{III.3.3})$$

(As one of the proofs, first choose an  $(n - k)$ -dimensional subspace  $V \subseteq \mathbb{F}_q^n$  as the image, and then choose a surjection  $\mathbb{F}_q^n \rightarrow V$ . The former has  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  choices, and the latter  $(q^n - 1)(q^n - q) \dots (q^n - q^{n-k-1})$  choices.)

Recalling that  $|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \dots (q^n - q^{n-1})$ , it follows (after simplification) that

$$\widehat{Z}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{|\text{GL}_n(\mathbb{F}_q)|} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q (q^n - 1)(q^n - q) \dots (q^n - q^{n-k-1}) x^n \quad (\text{III.3.4})$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{\begin{bmatrix} n \\ k \end{bmatrix}_t}{(t; t)_k} x^n, \quad (\text{III.3.5})$$

where  $t = q^{-1}$ .

### III.3.2: Factorization of $\widehat{Z}(x)$

We collect some standard  $q$ -series identities that will be used in the factorization.

**Proposition III.3.1** ([And98]). *As formal power series in  $t$  (and  $x$  if applicable), we have*

(a)  $\sum_{\ell(\lambda) \leq k} t^{|\lambda|} = \frac{1}{(t; t)_k}$ . *Here the sum is over all partitions with at most  $k$  parts.*

(b)  $\sum_{\lambda \subseteq (n-k) \times k} t^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_t$ . *The notation here means that the sum is over all partitions whose Young diagram fits inside a  $(n - k) \times k$  rectangle.*

$$(c) \sum_{n=0}^{\infty} \frac{x^n}{(t; t)_n} = \frac{1}{(x; t)_{\infty}}.$$

(d) A partition with zeros is a nonincreasing sequence of finitely many nonnegative integers  $(\lambda_1, \lambda_2, \dots, \lambda_{\ell})$ , whose length is defined as  $\ell$ . Then we have the identity

$$\sum_{\substack{\lambda \text{ w/ } [0] \\ \lambda_1 \leq k}} t^{|\lambda|} x^{\ell(\lambda)} = \frac{1}{(1-x)(1-tx) \dots (1-t^k x)}, \quad (\text{III.3.6})$$

where the sum is over all partitions with zeros whose parts have sizes at most  $k$ .

**Lemma III.3.2.** *We have*

$$\widehat{Z}(x) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{[n]_t}{(t; t)_k} x^n = \sum_{\lambda \text{ w/ } [0]} t^{|\lambda| - \sigma_1(\lambda)^2} x^{\ell(\lambda)} \quad (\text{III.3.7})$$

*Proof.* A partition  $\lambda$  whose Durfee square has sidelength  $k$  can be reconstructed uniquely with a partition  $\lambda^{(1)}$  such that  $\ell(\lambda^{(1)}) \leq k$  (to be put to the right of the Durfee square) and a partition  $\lambda'$  such that  $\lambda'_1 \leq k$  (to be put below the Durfee square).

By Proposition III.3.1(1)(2), we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{[n]_t}{(t; t)_k} x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^n \left( \sum_{\lambda^{(1)} \subseteq k \times \infty} t^{|\lambda^{(1)}|} \sum_{\lambda'_1 \subseteq (n-k) \times k} t^{|\lambda'|} \right) \quad (\text{III.3.8})$$

$$= \sum_{n=0}^{\infty} x^n \sum_{\lambda \subseteq n \times \infty} t^{|\lambda| - \sigma_1(\lambda)^2} \quad (\text{III.3.9})$$

$$= \sum_{n=0}^{\infty} x^n \sum_{\substack{\lambda \text{ w/ } [0] \\ \ell(\lambda) = n}} t^{|\lambda| - \sigma_1(\lambda)^2} \quad (\text{III.3.10})$$

$$= \sum_{\lambda \text{ w/ } [0]} t^{|\lambda| - \sigma_1(\lambda)^2} x^{\ell(\lambda)}. \quad (\text{III.3.11})$$

Here, the line (III.3.9) is because  $\lambda^{(1)}$  and  $\lambda'$  consist of the part of  $\lambda$  outside the Durfee square. Note that  $\lambda \subseteq n \times \infty$  means that  $\lambda$  fits inside the  $n$  (rows)  $\times \infty$  (columns) rectangle, which is equivalent to saying  $\ell(\lambda) \leq n$ . The line (III.3.10) is because specifying a partition whose length is at most  $n$  is equivalent to specifying a partition with zeros whose length is exactly  $n$ .  $\square$

To complete the factorization, we reconstruct  $\lambda$  with zeros using the first two Durfee squares. Pick  $k \geq l \geq 0$ . Let  $\lambda^{(1)} \subseteq k \times \infty$  and  $\lambda^{(2)} \subseteq l \times (k-l)$  be usual partitions, and

$\lambda'' \subseteq \infty \times l$  be a partition with zeros. Then we have a bijection

$$\{(\lambda'' \text{ w/ } [0], \lambda^{(1)}, \lambda^{(2)}) \text{ as above}\} \rightarrow \{\lambda \text{ w/ } [0] : \sigma_1(\lambda) = k, \sigma_2(\lambda) = l\} \quad (\text{III.3.12})$$

by putting  $\lambda^{(1)}$  to the right of the first Durfee square,  $\lambda^{(2)}$  to the right of the second Durfee square, and  $\lambda''$  below the second Durfee square.

We have

$$\widehat{Z}(x) = \sum_{\lambda \text{ w/ } [0]} t^{|\lambda| - \sigma_1(\lambda)^2} x^{\ell(\lambda)} \quad (\text{III.3.13})$$

$$= \sum_{\substack{k \geq l \geq 0 \\ \lambda'', \lambda^{(1)}, \lambda^{(2)}}} t^{|\lambda^{(1)}|^2 + |\lambda^{(2)}|^2 + |\lambda''|^2 + l^2} x^{k+l+\ell(\lambda'')} \quad (\text{III.3.14})$$

$$= \sum_{k \geq l \geq 0} t^{l^2} x^{l+k} \left( \sum_{\lambda^{(1)} \subseteq k \times \infty} t^{|\lambda^{(1)}|} \sum_{\lambda^{(2)} \subseteq k \times \infty} t^{|\lambda^{(2)}|} \sum_{\substack{\lambda'' \text{ w/ } [0] \\ \lambda'' \subseteq \infty \times l}} t^{|\lambda''|} x^{\ell(\lambda'')} \right) \quad (\text{III.3.15})$$

$$= \sum_{k \geq l \geq 0} t^{l^2} x^{l+k} \frac{1}{(t; t)_k} \left[ \begin{matrix} k \\ l \end{matrix} \right]_t \frac{1}{(1-x)(1-tx) \dots (1-t^l x)}, \quad (\text{III.3.16})$$

where the line (III.3.16) uses Proposition III.3.1(1)(2)(4), in that order.

Observe that

$$\frac{1}{(t; t)_k} \left[ \begin{matrix} k \\ l \end{matrix} \right]_t = \frac{1}{(t; t)_k} \frac{(t; t)_k}{(t; t)_l (t; t)_{k-l}} = \frac{1}{(t; t)_l (t; t)_{k-l}}. \quad (\text{III.3.17})$$

Letting  $b = k - l$ , we have

$$\widehat{Z}(x) = \sum_{k \geq l \geq 0} t^{l^2} x^{l+k} \frac{1}{(t; t)_k} \left[ \begin{matrix} k \\ l \end{matrix} \right]_t \frac{1}{(1-x)(1-tx) \dots (1-t^l x)} \quad (\text{III.3.18})$$

$$= \sum_{b, l \geq 0} t^{l^2} x^{l+(b+l)} \frac{1}{(t; t)_l (t; t)_b} \frac{1}{(1-x)(1-tx) \dots (1-t^l x)} \quad (\text{III.3.19})$$

$$= \left( \sum_{b=0}^{\infty} \frac{x^b}{(t; t)_b} \right) \left( \sum_{l=0}^{\infty} t^{l^2} x^{2l} \frac{1}{(t; t)_l} \frac{1}{(1-x)(1-tx) \dots (1-t^l x)} \right) \quad (\text{III.3.20})$$

$$= \frac{1}{(x; t)_{\infty}} \sum_{l=0}^{\infty} t^{l^2} x^{2l} \frac{1}{(t; t)_l} \frac{1}{(1-x)(1-tx) \dots (1-t^l x)}, \quad (\text{III.3.21})$$

where the line (III.3.21) follows from Proposition III.3.1(3).

We remark that the key reason why this factorization works is that (III.3.17) does not depend explicitly on  $k$  after simplification.

Finally, recalling that  $(x; t)_\infty = (1 - x)(1 - tx)(1 - t^2x) \dots$ , we get

$$\widehat{Z}(x) = \frac{1}{(x; t)_\infty} \sum_{l=0}^{\infty} t^{l^2} x^{2l} \frac{1}{(t; t)_l} \frac{1}{(1 - x)(1 - tx) \dots (1 - t^l x)} \quad (\text{III.3.22})$$

$$= \frac{1}{(x; t)_\infty^2} \sum_{l=0}^{\infty} t^{l^2} x^{2l} \frac{1}{(t; t)_l} (1 - t^{l+1}x)(1 - t^{l+2}x) \dots \quad (\text{III.3.23})$$

$$=: \frac{1}{(x; t)_\infty^2} H_q(x), \quad (\text{III.3.24})$$

which finishes the proof of Theorem III.1.2.

### III.4: General properties of the Cohen–Lenstra series

We give a self-contained introduction about the well-known properties of the Cohen–Lenstra series. These properties are implicit in [BM15, GZLMH04] from the motivic point of view, while in this introduction, we restrict our attention to counting over finite fields. We point out that the argument in Proposition III.4.5 that uses Proposition III.4.2 is essentially the use of “power structures” in [BM15, GZLMH04].

Let  $R$  be an algebra over  $\mathbb{F}_q$  with only finite quotient fields, and let  $X$  be a variety over  $\mathbb{F}_q$ . Recall the definitions

$$\widehat{Z}_R(x) := \sum_M \frac{1}{|\text{Aut } M|} x^{\dim_{\mathbb{F}_q} M}, \quad (\text{III.4.1})$$

where  $M$  ranges over all isomorphism classes of finite-cardinality  $R$ -modules, and

$$\widehat{Z}_X(x) := \sum_M \frac{1}{|\text{Aut } M|} x^{\dim_{\mathbb{F}_q} H^0(X; M)}, \quad (\text{III.4.2})$$

where  $M$  ranges over all isomorphism classes of finite-length coherent sheaves over  $X$ , and  $H^0(X; M)$  denotes the space of global sections of  $M$ . We denote both  $\dim_{\mathbb{F}_q} M$  and  $\dim_{\mathbb{F}_q} H^0(X; M)$  by  $\deg M$ . We also recall the local Cohen–Lenstra series for a closed point  $p$  of  $X$ :

$$\widehat{Z}_{X,p}(x) := \widehat{Z}_{\mathcal{O}_{X,p}}(x). \quad (\text{III.4.3})$$

We state some basic properties.

#### Proposition III.4.1.

- (a)  $\widehat{Z}_R(x) = \widehat{Z}_{\text{Spec } R}(x)$ .
- (b)  $\widehat{Z}_{\mathcal{O}_{X,p}}(x) = \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x)$ .

(c) We have

$$\widehat{Z}_{X,p}(x) := \sum_{M_p} \frac{1}{|\mathrm{Aut} M_p|} x^{\deg M_p}, \quad (\text{III.4.4})$$

where  $M_p$  ranges over all isomorphism classes of finite-length coherent sheaves over  $X$  that are supported at  $p$ .

*Proof.*

- (a) This follows from the standard correspondence between modules over  $R$  and quasicoherent sheaves over  $\mathrm{Spec} R$ .
- (b) This follows from the elementary fact that the classification of finite-length modules over a Noetherian local ring is the same as the classification of finite-length modules over its completion.
- (c) A coherent sheaf is supported at  $p$  is determined by its stalk at  $p$ , thus corresponds to a module over  $\mathcal{O}_{X,p}$ .

□

**Proposition III.4.2** (Euler product). *Let  $X$  be a variety over  $\mathbb{F}_q$ . Then*

$$\widehat{Z}_X(x) = \prod_{p \in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x), \quad (\text{III.4.5})$$

where  $\mathrm{cl}(X)$  is the set of closed points in  $X$ .

*Proof.* For every finite-length coherent sheaf  $M$  over  $X$ , we have a unique decomposition  $M = \bigoplus_{p \in \mathrm{cl}(X)} M_p$  into finite-length coherent sheaves  $M_p$  supported at  $p$ , with all but finitely many  $M_p$ 's being zero. For closed points  $p \neq q$  and sheaves  $M_p, M_q$  supported on  $p, q$  respectively, we have

$$\mathrm{Hom}_X(M_p, M_q) = 0. \quad (\text{III.4.6})$$

It follows that

$$\mathrm{Aut}(M_{p_1} \oplus \cdots \oplus M_{p_r}) \cong \mathrm{Aut}(M_{p_1}) \times \cdots \times \mathrm{Aut}(M_{p_r}). \quad (\text{III.4.7})$$

As a consequence,

$$\widehat{Z}_X(x) = \sum_M \frac{1}{|\mathrm{Aut} M|} x^{\deg M} \quad (\text{III.4.8})$$

$$= \sum_{(M_p: p \in \mathrm{cl}(X))} \frac{1}{|\prod_p \mathrm{Aut} M_p|} x^{\sum_p \deg M_p} \quad (\text{III.4.9})$$

$$= \prod_{p \in \mathrm{cl}(X)} \sum_{M_p} \frac{1}{|\mathrm{Aut} M_p|} x^{\deg M_p} \quad (\text{III.4.10})$$

$$= \prod_{p \in \mathrm{cl}(X)} \widehat{Z}_{X,p}(x). \quad (\text{III.4.11})$$

□

For any subvariety  $Y$  of  $X$ , if we set

$$\widehat{Z}_{X,Y}(x) := \prod_{p \in \mathrm{cl}(Y)} \widehat{Z}_{X,p}(x) = \sum_{\mathrm{supp} M \subseteq Y} \frac{1}{|\mathrm{Aut} M|} x^{\deg M}, \quad (\text{III.4.12})$$

then the Euler product gives

$$\widehat{Z}_X(x) = \widehat{Z}_U(x) \cdot \widehat{Z}_{X,Z}(x) \quad (\text{III.4.13})$$

for any open subvariety  $U \subseteq X$  and closed subvariety  $Z \subseteq X$  with  $X \setminus Z = U$ . This uses the fact that  $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$  for all  $p \in \mathrm{cl}(U)$ , so that

$$\widehat{Z}_U(x) = \widehat{Z}_{X,U}(x) \text{ for open } U \subseteq X. \quad (\text{III.4.14})$$

As a warning,  $\widehat{Z}_{X,Z}(x)$  is not equal to  $\widehat{Z}_Z(x)$ , because  $\mathcal{O}_{Z,p}$  and  $\mathcal{O}_{X,p}$  are not isomorphic in general. Thus, the  $\widehat{Z}$  construction is not motivic in the sense that  $\widehat{Z}_X(x) \neq \widehat{Z}_U(x) \cdot \widehat{Z}_Z(x)$ .

The following relates the Cohen–Lenstra zeta function to the variety of modules, or its nilpotent variant.

**Proposition III.4.3.** *Let  $R = \frac{\mathbb{F}_q[t_1, \dots, t_m]}{(f_1, \dots, f_r)}$ , where  $t_1, \dots, t_m$  are indeterminates and  $f_1, \dots, f_r$  are polynomials in  $t_1, \dots, t_m$ . Then*

(a) *We have*

$$\widehat{Z}_R(x) = \sum_{n=0}^{\infty} \frac{|M_n|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n, \quad (\text{III.4.15})$$

where

$$M_n := \left\{ (A_1, \dots, A_m) \left| \begin{array}{l} A_i \in \mathrm{Mat}_n(\mathbb{F}_q), A_i A_j = A_j A_i \\ f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \leq s \leq r \end{array} \right. \right\} \quad (\text{III.4.16})$$



(b) If  $f_1, \dots, f_r$  all vanish at the origin  $0$ , let  $p = (t_1, \dots, t_m)R$  be the maximal ideal corresponding to  $0 \in X := \text{Spec } R$ , then

$$\widehat{Z}_{R_p}(x) = \widehat{Z}_{X,0}(x) = \sum_{n=0}^{\infty} \frac{|N_n|}{|\text{GL}_n(\mathbb{F}_q)|} x^n, \quad (\text{III.4.17})$$

where

$$N_n := \left\{ (A_1, \dots, A_m) \left| \begin{array}{l} A_i \in \text{NilP}_n(\mathbb{F}_q), A_i A_j = A_j A_i \\ f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \leq s \leq r \end{array} \right. \right\} \quad (\text{III.4.18})$$

(c) More generally, let  $I \subseteq J \subseteq \mathbb{F}_q[t_1, \dots, t_m]$  be two ideals, and  $Z = \text{Spec } \mathbb{F}_q[t_1, \dots, t_m]/J \subseteq X = \text{Spec } \mathbb{F}_q[t_1, \dots, t_m]/I$ . Then

$$\widehat{Z}_{X,Z}(x) = \sum_{n=0}^{\infty} \frac{|N_n|}{|\text{GL}_n(\mathbb{F}_q)|} x^n, \quad (\text{III.4.19})$$

where

$$N_n := \left\{ A = (A_1, \dots, A_m) \left| \begin{array}{l} A_i \in \text{Mat}_n(\mathbb{F}_q), A_i A_j = A_j A_i \\ f(A) = 0 \text{ for } f \in I, g(A) \in \text{NilP}_n(\mathbb{F}_q) \text{ for } g \in J \end{array} \right. \right\} \quad (\text{III.4.20})$$

*Proof.*

(a) Fix  $n$  and consider an  $\mathbb{F}_q$ -vector space  $V$  of dimension  $n$ . Giving  $V$  a structure of an  $R$ -module is equivalent to specifying the actions of  $t_1, \dots, t_m$  on  $V$  as linear endomorphisms  $A_1, \dots, A_m$ , under the constraints

$$A_i A_j = A_j A_i \quad (\text{III.4.21})$$

and

$$f_s(A_1, \dots, A_m) = 0 \text{ for } 1 \leq s \leq r. \quad (\text{III.4.22})$$

We denote by  $(V; A_1, \dots, A_m)$  the  $R$ -module specified by the data above. Note that the constraints are satisfied if and only if  $(A_1, \dots, A_m) \in M_n$ .

Consider the action of  $GL_n(\mathbb{F}_q)$  on the set  $M_n$  by simultaneous conjugation:

$$g \cdot (A_1, \dots, A_m) := (g A_1 g^{-1}, \dots, g A_m g^{-1}). \quad (\text{III.4.23})$$

Given two  $R$ -modules  $M = (V; A_1, \dots, A_m)$  and  $M' = (V'; A'_1, \dots, A'_m)$ , an  $R$ -linear map  $M \rightarrow M'$  is nothing but an  $\mathbb{F}_q$ -linear map  $B : V \rightarrow V'$  such that  $B \circ A_i = A'_i \circ B$  for all  $i$ . It follows that  $(\mathbb{F}_q^n; A_1, \dots, A_m)$  and  $(\mathbb{F}_q^n; A'_1, \dots, A'_m)$  are isomorphic as  $R$ -modules precisely if  $(A_1, \dots, A_m)$  and  $(A'_1, \dots, A'_m)$  are in the same orbit. Moreover,  $g : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$  gives an automorphism of  $(\mathbb{F}_q^n; A_1, \dots, A_m)$  as an  $R$ -module if and only if  $g$  fixes  $(A_1, \dots, A_m)$ . Denoting by  $O_M$  the orbit corresponding to an  $R$ -module  $M$  with  $\dim_{\mathbb{F}_q} M = n$ , the orbit-stabilizer theorem gives

$$\sum_{\deg M=n} \frac{1}{|\text{Aut } M|} = \sum_M \frac{|O_M|}{|\text{GL}_n(\mathbb{F}_q)|} = \frac{|M_n|}{|\text{GL}_n(\mathbb{F}_q)|}, \quad (\text{III.4.24})$$

where the first and second sums are over the isomorphism classes of  $R$ -modules of degree  $M$ .

It follows that

$$\widehat{Z}_R(x) = \sum_{n=0}^{\infty} \sum_{\deg M=n} \frac{1}{|\text{Aut } M|} x^n = \sum_{n=0}^{\infty} \frac{|M_n|}{|\text{GL}_n(\mathbb{F}_q)|} x^n. \quad (\text{III.4.25})$$

(b) The proof is exactly the same, modulo the following observation:

*The category of finite-length  $R_p$ -modules is a full subcategory of finite-length  $R$ -modules consisting of those annihilated by some power of  $p$ .*

An  $R$ -module  $(V; A_1, \dots, A_m)$  is annihilated by a power of  $p = (t_1, \dots, t_m)R$  if and only if a power of  $A_i$  annihilates  $V$  for all  $i$ . This is equivalent to requiring that all  $A_i$  are nilpotent.

(c) Let  $R = \text{Spec } \mathbb{F}_q[t_1, \dots, t_m]/I$  and  $S = \text{Spec } \mathbb{F}_q[t_1, \dots, t_m]/J$ . It suffices to prove the following claim:

*A finite-length  $R$ -module  $M$  is supported on  $Z = \text{Spec } S$  if and only if  $J^n M = 0$  for some  $n$ .*

To prove the claim, assume  $J^n M = 0$ . Consider any maximal ideal  $p$  of  $R$  that corresponds to a closed point in  $X \setminus Z$ , then  $p$  does not contain  $J$ . So there exists  $u \in J$  such that  $u \notin p$ . Since  $u^n M = 0$  and  $u$  is invertible in  $R_p$ , the localization  $M_p$  of  $M$  at  $p$  is zero. This shows that  $M$  is supported on  $Z$ .

Conversely, assume  $M$  is supported on  $Z$ . Then there are maximal ideals  $p_1, \dots, p_h$

corresponding to closed points in  $Z$  such that

$$M = \bigoplus_{i=1}^h M_{p_i}, \quad (\text{III.4.26})$$

where  $M_{p_i}$  is the localization of  $M$  at  $p_i$ . Note that  $p_i \supseteq J$ . Since  $M_{p_i}$  is a finite-length module over the local ring  $R_{p_i}$ , there is  $n_i$  such that  $p_i^{n_i} M_{p_i} = 0$ . It follows that  $J^{n_i} M_{p_i} = 0$ . Taking  $n = \max\{n_1, \dots, n_h\}$ , we get  $J^n M = \bigoplus_{i=1}^h J^n M_{p_i} = 0$ , proving the claim. □

**Corollary III.4.4.**

$$\widehat{Z}_{\mathbb{F}_q[[t]]}(x) = \prod_{i=1}^{\infty} (1 - q^{-i}x)^{-1}, \quad (\text{III.4.27})$$

$$\widehat{Z}_{\mathbb{F}_q[[u,v]]}(x) = \prod_{i,j \geq 1} (1 - q^{-j}x^i)^{-1}. \quad (\text{III.4.28})$$

*Proof.* Given Proposition III.4.3(2), the two formulas follow from the matrix counting results of Fine and Herstein [FH58] and Fulman and Guralnick [FG18], respectively. □

**Proposition III.4.5.** *Let  $Z_X(x)$  be the Hasse–Weil zeta series of  $X$ . Then*

(a) *If  $X$  is a smooth curve over  $\mathbb{F}_q$ , then*

$$\widehat{Z}_X(x) = \prod_{i=1}^{\infty} Z_X(q^{-i}x) \in \mathbb{C}[[x]]. \quad (\text{III.4.29})$$

(b) *If  $X$  is a smooth surface over  $\mathbb{F}_q$ , then*

$$\widehat{Z}_X(x) = \prod_{i,j \geq 1} Z_X(q^{-j}x^i) \in \mathbb{C}[[x]]. \quad (\text{III.4.30})$$

*Proof.* (a) The Euler product (Proposition III.4.2) gives

$$\widehat{Z}_X(x) = \prod_{p \in \text{cl}(X)} \widehat{Z}_{X,p}(x) = \prod_{p \in \text{cl}(X)} \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x). \quad (\text{III.4.31})$$

Since  $X$  is a smooth curve,  $\widehat{\mathcal{O}}_{X,p}$  is a complete regular local ring of dimension one. By the Cohen structure theorem,  $\widehat{\mathcal{O}}_{X,p} \cong \kappa_p[[t]]$ , where  $\kappa_p$  is the residue field of  $\widehat{\mathcal{O}}_{X,p}$ . By

Corollary III.4.4,

$$\widehat{Z}_{\kappa_p[[t]]/\kappa_p}(x) = \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut} M|} x^{\dim_{\kappa_p} M} = \prod_{i=1}^{\infty} (1 - q^{-i \deg p} x)^{-1}, \quad (\text{III.4.32})$$

where  $\deg p$  is the degree of the field extension  $[\kappa_p : \mathbb{F}_q]$ , so that  $q^{\deg p}$  is the cardinality of  $\kappa_p$ .

Noting that  $\dim_{\mathbb{F}_q} M = (\deg p) \dim_{\kappa_p} M$  for any  $\kappa_p$ -vector space  $M$ , we have

$$\widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) = \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut} M|} x^{\dim_{\mathbb{F}_q} M} \quad (\text{III.4.33})$$

$$= \sum_{M/\kappa_p[[t]]} \frac{1}{|\mathrm{Aut} M|} (x^{\deg p})^{\dim_{\kappa_p} M} \quad (\text{III.4.34})$$

$$= \prod_{i=1}^{\infty} (1 - q^{-i \deg p} x^{\deg p})^{-1}. \quad (\text{III.4.35})$$

Recalling the Euler product of the Hasse–Weil zeta function

$$Z_X(x) = \prod_{p \in \mathrm{cl}(X)} (1 - x^{\deg p})^{-1}, \quad (\text{III.4.36})$$

we get

$$\widehat{Z}_X(x) = \prod_{p \in \mathrm{cl}(X)} \prod_{i=1}^{\infty} (1 - q^{-i \deg p} x^{\deg p})^{-1} \quad (\text{III.4.37})$$

$$= \prod_{i=1}^{\infty} \prod_{p \in \mathrm{cl}(X)} (1 - (q^{-i} x)^{\deg p})^{-1} \quad (\text{III.4.38})$$

$$= \prod_{i=1}^{\infty} Z_X(q^{-i} x). \quad (\text{III.4.39})$$

- (b) By the Cohen structure theorem, for any closed point  $p$  on a smooth surface  $p$ , we have  $\widehat{\mathcal{O}}_{X,p} \cong \kappa_p[[u, v]]$ . Applying the same argument to the corresponding formula in

Corollary III.4.4, we have

$$\widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) = \widehat{Z}_{\kappa_p[[u,v]]/\kappa_p}(x^{\deg p}) \quad (\text{III.4.40})$$

$$= \left( \prod_{i,j \geq 1} (1 - (q^{\deg p})^{-j} x^i)^{-1} \right) \Big|_{x \mapsto x^{\deg p}} \quad (\text{III.4.41})$$

$$= \prod_{i,j \geq 1} (1 - (q^{-j} x^i)^{\deg p})^{-1}. \quad (\text{III.4.42})$$

It follows that

$$\widehat{Z}_X(x) = \prod_{p \in \text{cl}(X)} \widehat{Z}_{\widehat{\mathcal{O}}_{X,p}}(x) \quad (\text{III.4.43})$$

$$= \prod_{i,j \geq 1} \prod_{p \in \text{cl}(X)} (1 - (q^{-j} x^i)^{\deg p})^{-1} \quad (\text{III.4.44})$$

$$= \prod_{i,j \geq 1} Z_X(q^{-j} x^i). \quad (\text{III.4.45})$$

□

### III.5: Properties of $H_q(x)$ and Proof of Theorem III.1.1

Theorem III.1.1(b) almost follow immediately from Theorem III.1.2: let  $X = \text{Spec } \mathbb{F}_q[u, v]/(uv)$  be the union of  $x$ - and  $y$ -axes on a plane, and let  $p$  be the origin, then by (III.4.13),

$$\widehat{Z}_{X,p}(x) = \frac{\widehat{Z}_X(x)}{\widehat{Z}_{X \setminus p}(x)} \quad (\text{III.5.1})$$

$$= \frac{(x; q^{-1})_{\infty}^{-2} H_q(x)}{(\widehat{Z}_{\mathbb{A}^1 \setminus 0}(x))^2} \quad (\text{III.5.2})$$

$$= \frac{(x; q^{-1})_{\infty}^{-2} H_q(x)}{(1-x)^{-2}} \quad (\text{III.5.3})$$

$$= (xq^{-1}; q^{-1})_{\infty}^{-2} H_q(x). \quad (\text{III.5.4})$$

Here,  $\mathbb{A}^1 \setminus 0$  denotes an affine line minus one point. Because it is a smooth curve, its Cohen–Lenstra series can be evaluated by Proposition III.4.5. This finishes the proof of Theorem III.1.1(b) except the claim that  $q^i, i \geq 1$  are actually double poles of  $\widehat{Z}_{\mathbb{F}_q[[u,v]]/(uv)}(x)$ . This requires that  $H_q(q^i) \neq 0$  for  $i \geq 1$ , which turns out to be elementary from the expression of  $H_q(x)$ , see Proposition III.5.1 below.

The proof of Theorem III.1.1 is complete given the observations about  $H_q(x)$  in Proposition III.5.1. Let  $t = q^{-1}$  and we define

$$H(x; t) := H_q(x) = \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t; t)_k} (1 - t^{k+1}x)(1 - t^{k+2}x) \dots \quad (\text{III.5.5})$$

$$= (tx; t)_{\infty} \sum_{k=0}^{\infty} t^{k^2} x^{2k} \frac{1}{(t; t)_k (tx; t)_k} \quad (\text{if } x \neq t^{-1}, t^{-2}, \dots). \quad (\text{III.5.6})$$

We note that the infinite sum (III.5.5) defines  $H(x; t)$  in two possible ways. First, the infinite sum converges formally to a power series in  $x$  and  $t$  (due to the  $x^{2k}$  factor). Second, if  $0 < t < 1$  is fixed, then each summand of (III.5.5) is an entire function in  $x$ , so (III.5.5) is an infinite sum of functions. We will show that the “formal” sum and the “analytic” sum are the same.

**Proposition III.5.1.** *For any fixed real number  $0 < t < 1$ , we have*

- (a) *The infinite sum (III.5.5) of entire functions in  $x$  converges uniformly on any bounded disc to an entire function whose Maclaurin series is the coefficient-wise limit of the sum (III.5.5) of formal power series in  $x$ .*
- (b)  *$H(x; t) > 0$  if  $x < t^{-1}$  or  $x = t^{-i}, i = 1, 2, \dots$*
- (c)  *$H(1; t) = 1$ .*
- (d)  *$H(-1; t) = (-t; t)_{\infty} (-t; t^2)_{\infty}$*

*Proof.*

- (a) Fix a bounded disc  $|x| \leq M$ . Then as  $k$  goes to infinity, the factor  $\frac{1}{(t; t)_k} (1 - t^{k+1}x)(1 - t^{k+2}x) \dots$  has a uniform bound for  $|x| \leq M$  that only depends on  $M$ . The uniform convergence of the sum (III.5.5) follows from the convergence of  $\sum t^{k^2} M^k$ . Therefore, the sum (III.5.5) defines an entire function.

To find its Maclaurin series, consider the sequence of partial sums of (III.5.5). The assertion of (a) follows from the fact that if a sequence of holomorphic functions  $f_k(x)$  converges uniformly to a holomorphic function  $f(x)$  on a disc  $D$  centered at  $x = 0$ , then the Maclaurin series of  $f_k(x)$  must converge to the Maclaurin series of  $f(x)$  coefficient-wise. For a proof, we recall that the  $n$ -th Maclaurin coefficient of  $f(x)$  is given by  $n!f^{(n)}(0)$ . For any  $n$ , the sequence  $f_k^{(n)}(x)$  converges uniformly to  $f^{(n)}(x)$  on compact subsets of  $D$  (see for instance [Rud87, Theorem 10.28]), so we have  $n!f_k^{(n)}(0) \rightarrow n!f^{(n)}(0)$  as  $k \rightarrow \infty$ .

(b) If  $x < t^{-1}$ , then every term of (III.5.5) is positive. If  $x = t^{-i}$ ,  $i = 1, 2, \dots$ , then

$$H(t^{-i}; t) = \sum_{k=0}^{\infty} \frac{t^{k^2} (t^{-i})^{2k}}{(1-t) \dots (1-t^k)} (1 - t^{k+1} t^{-i}) (1 - t^{k+2} t^{-i}) \dots \quad (\text{III.5.7})$$

$$= \sum_{k=i}^{\infty} \frac{t^{k^2} (t^{-i})^{2k}}{(1-t) \dots (1-t^k)} (1 - t^{k+1-i}) (1 - t^{k+2-i}) \dots \quad (\text{III.5.8})$$

and every term in the last sum is positive.

(c) Using (III.5.6), we have

$$H(1; t) = (t; t)_{\infty} \sum_{k=0}^{\infty} t^{k^2} \frac{1}{(t; t)_k (t; t)_k}, \quad (\text{III.5.9})$$

which is equal to 1 by the following standard identity due to Euler; see [And98, p. 21, (2.2.9)].

$$\sum_{k=0}^{\infty} \frac{t^{k^2}}{(t; t)_k^2} = \frac{1}{(t; t)_{\infty}}. \quad (\text{III.5.10})$$

(d) To compute  $H(-1; t)$ , we need the following identities:

$$(t^2; t^2)_n = (t; t)_n (-t; t)_n \quad (\text{III.5.11})$$

$$\sum_{n=0}^{\infty} \frac{t^{\binom{n}{2}} x^n}{(t; t)_n} = (-x; t)_{\infty}, \quad (\text{III.5.12})$$

where the first one is elementary:

$$(t^2; t^2)_n = \prod_{i=1}^n (1 - t^{2i}), \quad (\text{III.5.13})$$

$$= \prod_{i=1}^n (1 - t^i)(1 + t^i) = (t; t)_n (-t; t)_n, \quad (\text{III.5.14})$$

and the second one is due to Euler; see [And98, p. 19, (2.2.6)]. Now we have

$$H(-1; t) = (-t; t)_{\infty} \sum_{k=0}^{\infty} \frac{t^{k^2}}{(t; t)_k (-t; t)_k} \quad (\text{III.5.15})$$

$$= (-t; t)_{\infty} \sum_{k=0}^{\infty} \frac{(t^2)^{\binom{k}{2}} t^k}{(t^2; t^2)_k} \quad (\text{by (III.5.11)})$$

$$= (-t; t)_{\infty} (-t; t^2)_{\infty}. \quad (\text{by (III.5.12) with } x \mapsto t, t \mapsto t^2)$$

□

The proof of Theorem III.1.1 now follows directly from Theorem III.1.2 and Proposition III.5.1. Here, we note that the proof of Theorem III.1.1(c) requires the following computation:

$$\widehat{Z}_{\mathbb{F}_q[[u,v]]/(uv)}(-1) = (-t; t)_\infty^{-2} H(-1; t) \quad (\text{III.5.16})$$

$$= \frac{(-t; t^2)_\infty}{(-t; t)_\infty} \quad (\text{III.5.17})$$

$$= \frac{(1+t)(1+t^3)(1+t^5)\dots}{(1+t)(1+t^2)(1+t^3)\dots} \quad (\text{III.5.18})$$

$$= \frac{1}{(1+t^2)(1+t^4)(1+t^6)\dots} \quad (\text{III.5.19})$$

$$= \frac{1}{(-t^2; t^2)_\infty}. \quad (\text{III.5.20})$$

### III.6: Further discussion on $H_q(x)$

The factor  $H_q(x)$  that appears in the factorization of  $\widehat{Z}_{X,p}(x)$  (where  $(X, p)$  is a nodal singularity) is mostly mysterious. We note that if Question III.1.3(b) has a positive answer, then we can attach a meaningful holomorphic function  $H_{X,p}(x)$  to a curve singularity  $(X, p)$ . In particular,  $H_q(x)$  would be  $H_{X,p}(x)$  associated to a nodal singularity  $(X, p)$ . It is natural to ask about the properties of  $H_{X,p}(x)$  for a general singularity  $(X, p)$ , and what they reveal about the geometry of  $X$  at  $p$ . This section discusses possible analytic properties of  $H_q(x)$ , aiming at providing clues to the questions above.

The mysterious function  $H_q(x) = H(x; t)$  (where  $t = q^{-1} \in (0, 1/2]$  is fixed) appears to share some analytical features with the *partial theta function*  $\Theta_p(x; t) := \sum_{n=0}^{\infty} t^{n^2} x^n$ . We summarize several notable properties of the partial theta function; we refer the readers to an excellent survey paper [War] where many references are listed. The partial theta function satisfies the functional equation

$$\Theta_p(x; t) - tx\Theta_p(t^2x; t) = 1. \quad (\text{III.6.1})$$

An important property of the partial theta function is having *smooth coefficients*. An entire function  $f(x) = \sum a_n x^n$  is said to have smooth coefficients if  $\lim_{n \rightarrow \infty} a_n^2 / (a_{n-1} a_{n+1})$  converges. The partial theta function has smooth coefficients because  $a_n^2 / (a_{n-1} a_{n+1})$  is constant.

Having smooth coefficients is the reason behind many other analytic behaviors of  $\Theta_p(x; t)$ ,



such as distribution of roots in an “almost geometric sequence”, belong to the Laguerre–Pólya class, etc.; see [War] for an excellent survey paper on this topic. Therefore, having smooth coefficients is a key feature to look for when comparing  $H(x; t)$  to  $\Theta_p(x; t)$ .

Based on numerical observation, the roots of  $H(x; t)$  appear to be imaginary, and  $H(x; t)$  does not appear to have smooth coefficients. However, the even-degree terms and odd-degree terms of  $H(x; t)$  appear to have smooth coefficients and real roots. For any power series  $f(x) = \sum a_n x^n$ , denote

$$\ell_x f(x) := \lim_{n \rightarrow \infty} \frac{a_n^2}{a_{n-1} a_{n+1}}. \quad (\text{III.6.2})$$

Our observations suggest the following conjecture.

**Conjecture III.6.1.** The function  $H(x; t)$  satisfies the following properties:

(a) As a power series in  $x$  and  $t$ , we have

$$H(x; t) = \sum_{n=0}^{\infty} (-1)^n t^{\lceil n^2/4 \rceil} (1 + O(t)) x^n. \quad (\text{III.6.3})$$

(b) Let  $F(x; t)$  and  $G(x; t)$  be defined such that  $H(x; t) = F(x^2; t) + xG(x^2; t)$ . Then both  $F(x; t)$  and  $G(x; t)$  have smooth coefficients. Moreover, both  $\ell_x F(x; t)$  and  $\ell_x G(x; t)$  are equal to  $t^2$ .

**Question III.6.2.** Does  $F(x; t)$  (or  $G(x; t)$ ) satisfy a functional equation, possibly similar to (III.6.1), the functional equation for  $\Theta_p(x; t)$ ?

We note  $\ell_x \Theta_p(x; t) = t^2$ , the same as the conjectured value of  $\ell_x F(x; t)$  and  $\ell_x G(x; t)$ .

Apart from the similarity to the partial theta function, another motivation why we look for a functional equation for  $H(x; t)$  is an observation by Cohen and Lenstra [CL84, §7], where they find a functional equation for an entire function built from the Cohen–Lenstra zeta function of a Dedekind domain.

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# CHAPTER IV

## Counting on the Variety of Modules over the Quantum Plane

The content of this chapter is available at [\[Hua22\]](#).

### Abstract

Let  $\zeta$  be a fixed nonzero element in a finite field  $\mathbb{F}_q$  with  $q$  elements. In this article, we count the number of pairs  $(A, B)$  of  $n \times n$  matrices over  $\mathbb{F}_q$  satisfying  $AB = \zeta BA$  by giving a generating function. This generalizes a generating function of Feit and Fine that counts pairs of commuting matrices. Our result can be also viewed as the point count of the variety of modules over the quantum plane  $XY = \zeta YX$ , whose geometry was described by Chen and Lu.

### IV.1: Introduction

#### IV.1.1: Main results

Fix a nonzero element  $\zeta$  in  $\mathbb{F}_q$ , the finite field with  $q$  elements. Let  $\text{ord}(\zeta)$  denote the smallest positive integer  $m$  such that  $\zeta^m = 1$  in  $\mathbb{F}_q$ . We define the set of  $\mathbb{F}_q$ -points of the  $\zeta$ -commuting variety to be

$$K_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA\}. \quad (\text{IV.1.1})$$

The  $\zeta$ -commuting variety  $K_{\zeta,n}$  can be viewed as the variety of  $n$ -dimensional modules over the algebra of the quantum plane, namely, the noncommutative associative algebra in variables  $X$  and  $Y$  such that  $XY = \zeta YX$ . The geometry of the  $\zeta$ -commuting variety was studied by Chen and Lu [\[CL19\]](#), where explicit descriptions of its irreducible components and of a GIT quotient were given. The combinatorics of the  $\zeta$ -commuting has also been

studied when  $\zeta = 1$ : Feit and Fine [FF60] gave an explicit formula for the point count of the *commuting* variety (namely,  $K_{1,n}$ ) over a finite field, and Bryan and Morrison [BM15] proved that the “same” formula computes the motivic class of the commuting variety (over  $\mathbb{C}$ ) in the Grothendieck ring of varieties.

The focus of this paper is to count the cardinality of  $K_{\zeta,n}(\mathbb{F}_q)$  for  $\zeta$  in general. As a special case, the cardinality of  $K_{1,n}(\mathbb{F}_q)$ , the set of pairs of commuting matrices, was determined by Feit and Fine [FF60] in the form of a generating function. We give a generating function for  $|K_{\zeta,n}(\mathbb{F}_q)|$  that generalizes the  $\zeta = 1$  case.

**Theorem IV.1.1.** *Let  $m = \text{ord}(\zeta)$ ; in other words,  $\zeta$  is a primitive  $m$ -th root of unity of  $\mathbb{F}_q$ . We have the following identity of power series in  $x$ :*

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})} x^n = \prod_{i=1}^{\infty} F_m(x^i; q), \quad (\text{IV.1.2})$$

where

$$F_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)} \cdot \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \cdots}. \quad (\text{IV.1.3})$$

We note that  $|K_{\zeta,n}(\mathbb{F}_q)|$  only depends on the order  $m$  of  $\zeta$ . When  $m = 1$ , we recover the generating function given by Feit and Fine.

Theorem IV.1.1 is a direct consequence of the following result, which in itself can be viewed as a refinement of Theorem IV.1.1. We define

$$U_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA, A \text{ nonsingular}\}, \quad (\text{IV.1.4})$$

and

$$N_{\zeta,n}(\mathbb{F}_q) := \{(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q) : AB = \zeta BA, A \text{ nilpotent}\}. \quad (\text{IV.1.5})$$

When  $\zeta = -1$ , the variety  $N_{-1,n}$  is the semi-nilpotent anti-commuting variety, whose irreducible components and their dimensions are explicitly described by Chen and Wang [CW20].

For brevity reason, we put  $|\text{GL}_n(\mathbb{F}_q)|$  in place of  $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$  in the formulas below, noting that  $|\text{GL}_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$ .

**Theorem IV.1.2.** *Let  $m = \text{ord}(\zeta)$ . We have the following identities of power series in  $x$ :*

$$(a) \quad \sum_{n=0}^{\infty} \frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \left( \sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \left( \sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n \right) \quad (\text{IV.1.6})$$

(b)

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} G_m(x^i; q), \quad (\text{IV.1.7})$$

where

$$G_m(x; q) := \frac{1 - x^m}{(1 - x)(1 - x^m q)}. \quad (\text{IV.1.8})$$

(c)

$$\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} H(x^i; q), \quad (\text{IV.1.9})$$

where

$$H(x; q) := \frac{1}{(1 - x)(1 - xq^{-1})(1 - xq^{-2}) \dots}. \quad (\text{IV.1.10})$$

Using Theorem IV.1.2(a), Theorem IV.1.1 follows from the observation  $F_m(x; q) = G_m(x; q)H(x; q)$ .

Note that Theorem IV.1.2(c) implies that  $|N_{\zeta,n}(\mathbb{F}_q)|$  does not depend on  $m$  or  $\zeta$ , as long as  $\zeta \neq 0$ . In particular,  $|N_{\zeta,n}(\mathbb{F}_q)|$  always equals  $|N_{1,n}(\mathbb{F}_q)|$ , which is known to Feit and Fine. Therefore, the nontrivial dependence of  $|K_{\zeta,n}(\mathbb{F}_q)|$  on  $\zeta$  stems purely from that of  $|U_{\zeta,n}(\mathbb{F}_q)|$ .

### IV.1.2: History and related work

An important starting case in the study of varieties of modules is the commuting variety  $K_{1,n} = \{(A, B) : A, B \in \mathrm{Mat}_n, AB = BA\}$ . The commuting variety over  $\mathbb{C}$  was shown to be irreducible by Gerstenhaber [Ger61] and Motzkin and Taussky [MT55]. Its point count was given by Feit and Fine [FF60]. This result was reproved by Bryan and Morrison [BM15] from the perspective of motivic Donaldson–Thomas theory.

The commuting variety can be viewed in the context of Lie algebras. Let  $(\mathfrak{g}, [\cdot, \cdot])$  be a Lie algebra over an algebraically closed field. Define the *commuting variety* of  $\mathfrak{g}$  as

$$C(\mathfrak{g}) := \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : [x, y] = 0\}, \quad (\text{IV.1.11})$$

then  $K_{1,n}$  is the commuting variety of the Lie algebra of  $n$  by  $n$  matrices. As a generalization of the irreducibility result of  $K_{1,n}$ , Richardson [Ric79] showed that the commuting variety of any reductive Lie algebra over  $\mathbb{C}$  is irreducible. Levy [Lev02] extended this result to positive characteristic under mild restrictions on the Lie algebra. On the combinatorics side, Fulman and Guralnick [FG18] generalized the point-count result of Feit and Fine to commuting varieties of unitary groups and of odd characteristic symplectic groups. We also point out some papers that relate counting problems in Lie algebras to maximal tori of Lie groups; see

[FJRW17] and [Leh92b].

The focus of this paper, the  $\zeta$ -commuting variety  $K_{\zeta,n}$ , is another generalization of the commuting variety  $K_{1,n}$ . When  $\zeta = -1$ , we get the anti-commuting variety, whose geometry over  $\mathbb{C}$  was studied by Chen and Wang [CW20]. They gave explicit descriptions of the irreducible components of  $K_{-1,n}$  and of several variants. The above work was extended to general  $\zeta$  by Chen and Lu [CL19]. It is worth noting that  $K_{\zeta,n}$  is not irreducible unless  $\zeta = 1$ . The main contribution of our paper is the point count of  $K_{\zeta,n}$ .

The point count of  $K_{\zeta,n}$  can also be viewed as statistical information on the classification of modules over the quantum plane. In specific, since an  $n$ -dimensional<sup>1</sup> module over the quantum plane  $\mathbb{F}_q\{X, Y\}/(XY - \zeta YX)$  can be parametrized by a pair of matrices  $(A, B)$  in  $K_{\zeta,n}$ , the standard orbit-stabilizer argument gives

$$\frac{|K_{\zeta,n}(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|} = \sum_{\dim M=n} \frac{1}{|\mathrm{Aut} M|}, \quad (\text{IV.1.12})$$

where  $M$  ranges over all isomorphism classes of  $n$ -dimensional modules over the quantum plane. In other words, the  $x^n$ -coefficient of the generating function in (IV.1.1) is the weighted count of isomorphism classes of  $n$ -dimensional modules over the quantum plane, with weight inversely proportional to the size of the automorphism group (this weighting is commonly known as the *Cohen–Lenstra measure*, following the important work of Cohen and Lenstra [CL84] on random abelian groups). While Theorem IV.1.1 neither requires nor gives a classification of finite-dimensional modules, it does compute their total weight. It is unknown whether Theorem IV.1.1 can be verified using a classification, via the interpretation (IV.1.12). For work towards the classification of finite-dimensional modules over the quantum plane, we refer the reader to Bavula [Bav97, §3], where a classification of *simple* modules are given.

For a fixed integer  $g \geq 1$ , Hausel and Rodriguez-Villegas studied a related counting problem [HRV08, Eq (3.2.3)]

$$N_n(q) := |\{A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}_n(\mathbb{F}_q) : [A_1, B_1] \dots [A_g, B_g] \zeta_n = 1\}|, \quad (\text{IV.1.13})$$

where  $[A, B] := ABA^{-1}B^{-1}$  and  $\zeta_n$  is a primitive  $n$ -th root of unity of  $\mathbb{F}_q$ . If  $g = 1$ , then the defining equation for  $N_n(q)$  is  $A_1 B_1 = \zeta_n B_1 A_1$  (replacing  $\zeta_n^{-1}$  by  $\zeta_n$  in the process), so we have

$$N_n(q) = |K_{\zeta_n, n}^{\mathrm{GL} \times \mathrm{GL}}(\mathbb{F}_q)| \quad (\text{IV.1.14})$$

in the notation of Remark IV.2.2. We emphasize that  $N_n(q)$  are the *diagonal* entries of

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<sup>1</sup>The dimensionality refers to the dimension as an  $\mathbb{F}_q$ -vector space.

the table  $|K_{\zeta_m, n}^{\text{GL} \times \text{GL}}(\mathbb{F}_q)|$  in  $m, n$ , which we determine in (IV.3.21) in terms of a generating function.

Hausel and Rodriguez-Villegas observed a curious functional equation [HRV08, Eq (3.5.12)] about a generating function of  $N_n(q)$  that holds for all  $g$ , which reads

$$[x^n]E_{\zeta_n}^{\text{GL} \times \text{GL}}(x; q) = -q[x^n]E_{\zeta_n}^{\text{GL} \times \text{GL}}(x; q^{-1}) \quad (\text{IV.1.15})$$

when  $g = 1$ , where  $E_{\zeta_m}^{\text{GL} \times \text{GL}}(x; q)$  is a generating function of  $K_{\zeta_m, n}^{\text{GL} \times \text{GL}}$  defined in Remark IV.2.2, and the operator  $[x^n]$  refers to extracting the  $x^n$ -coefficient. From our formula (IV.3.21) for  $E_{\zeta_m}^{\text{GL} \times \text{GL}}(x; q)$ , we have

$$[x^n]E_{\zeta_n}^{\text{GL} \times \text{GL}}(x; q) = q - 1, \quad (\text{IV.1.16})$$

so the  $g = 1$  case of the functional equation reads

$$q - 1 = -q(q^{-1} - 1). \quad (\text{IV.1.17})$$

## IV.2: Proof of Theorem IV.1.2(a)

We recall that Theorem IV.1.2(a) claims that  $|K_{\zeta, n}(\mathbb{F}_q)|$  for all  $n$  can be recovered from  $|U_{\zeta, n}(\mathbb{F}_q)|$  and  $|N_{\zeta, n}(\mathbb{F}_q)|$  for all  $n$ . We start by proving a decomposition lemma, following the approach of Feit and Fine [FF60].

Let  $V$  be an  $n$ -dimensional vector space over any field, then by Fitting's lemma (see for instance [Jac89, p. 113]), for any linear map  $A \in \text{End}(V)$ , there is a unique decomposition  $V = K_A \oplus I_A$  such that  $A(K_A) \subseteq K_A$ ,  $A(I_A) \subseteq I_A$ ,  $A|_{K_A}$  is nilpotent, and  $A|_{I_A}$  is nonsingular.

**Lemma IV.2.1.** *Fix a linear map  $A \in \text{End}(V)$  and a nonzero scalar  $\zeta$ . Then a linear map  $B \in \text{End}(V)$  satisfies  $AB = \zeta BA$  if and only if*

- (a)  $B(K_A) \subseteq K_A$ ,  $B(I_A) \subseteq I_A$ .
- (b)  $A|_{K_A}B|_{K_A} = \zeta B|_{K_A}A|_{K_A}$ ,  $A|_{I_A}B|_{I_A} = \zeta B|_{I_A}A|_{I_A}$ .

*Proof.* Having the decomposition  $V = K_A \oplus I_A$ , any linear map  $X \in \text{End}(V)$  can be written as a matrix

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad \begin{array}{l} X_1 \in \text{End}(K_A), X_2 \in \text{Hom}(I_A, K_A), \\ X_3 \in \text{Hom}(K_A, I_A), X_4 \in \text{End}(I_A). \end{array} \quad (\text{IV.2.1})$$

Then we have

$$A = \begin{bmatrix} N & 0 \\ 0 & U \end{bmatrix} \quad (\text{IV.2.2})$$

where  $N \in \text{End}(K_A)$  is nilpotent and  $U \in \text{End}(I_A)$  is nonsingular. For an arbitrary  $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ , the equation  $AB = \zeta BA$  is equivalent to

$$\begin{cases} NB_1 = \zeta B_1 N, \\ NB_2 = \zeta B_2 U, \\ UB_3 = \zeta B_3 N, \\ UB_4 = \zeta B_4 U. \end{cases} \quad (\text{IV.2.3})$$

We note that  $B_2$  must be zero. Suppose not, since  $N$  is nilpotent, there exists an integer  $r \geq 0$  such that  $N^r B_2 \neq 0$  but  $N^{r+1} B_2 = 0$ . The second equation gives  $N^{r+1} B_2 = \zeta N^r B_2 U$ . The left-hand side is zero, while the right-hand side is nonzero because  $\zeta$  is a nonzero scalar and  $U$  is nonsingular. This yields a contradiction.

A similar argument shows that  $B_3 = 0$ , completing the proof of the lemma.  $\square$

Let  $V = \mathbb{F}_q^n$ . To choose  $A, B \in \text{End}(V)$  with  $AB = \zeta BA$ , because of Lemma IV.2.1, it suffices to choose a decomposition  $V = K \oplus I$ , and then choose  $A_K, B_K \in \text{End}(K)$ ,  $A_I, B_I \in \text{End}(I)$  such that  $A_K$  is nilpotent,  $A_K B_K = \zeta B_K A_K$ ,  $A_I$  is nonsingular, and  $A_I B_I = \zeta B_I A_I$ . We have

$$|K_{\zeta, n}(\mathbb{F}_q)| = \sum_{s+t=n} h(s, t) |N_{\zeta, s}(\mathbb{F}_q)| |U_{\zeta, t}(\mathbb{F}_q)|, \quad (\text{IV.2.4})$$

where  $h(s, t)$  is the number of ordered pairs  $(K, I)$  of subspaces of  $V$  such that  $\dim K = s$ ,  $\dim I = t$ .

It is noted by Feit and Fine [FF60, Equation (5)] that

$$h(s, t) = \frac{|\text{GL}_n(\mathbb{F}_q)|}{|\text{GL}_s(\mathbb{F}_q)| |\text{GL}_t(\mathbb{F}_q)|}. \quad (\text{IV.2.5})$$

It follows that

$$\sum_{n=0}^{\infty} \frac{|K_{\zeta, n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \sum_{n=0}^{\infty} \sum_{s+t=n} \frac{|\text{GL}_n(\mathbb{F}_q)|}{|\text{GL}_s(\mathbb{F}_q)| |\text{GL}_t(\mathbb{F}_q)|} |N_{\zeta, s}(\mathbb{F}_q)| |U_{\zeta, t}(\mathbb{F}_q)| \frac{1}{|\text{GL}_n(\mathbb{F}_q)|} x^n \quad (\text{IV.2.6})$$

$$= \sum_{s, t \geq 0} \frac{|N_{\zeta, s}(\mathbb{F}_q)|}{|\text{GL}_s(\mathbb{F}_q)|} \frac{|U_{\zeta, t}(\mathbb{F}_q)|}{|\text{GL}_t(\mathbb{F}_q)|} x^{s+t} \quad (\text{IV.2.7})$$

$$= \left( \sum_{s=0}^{\infty} \frac{|N_{\zeta, s}(\mathbb{F}_q)|}{|\text{GL}_s(\mathbb{F}_q)|} x^s \right) \left( \sum_{t=0}^{\infty} \frac{|U_{\zeta, t}(\mathbb{F}_q)|}{|\text{GL}_t(\mathbb{F}_q)|} x^t \right), \quad (\text{IV.2.8})$$

completing the proof of Theorem IV.1.2(a).

*Remark IV.2.2.* The same argument can prove two other similar factorization identities below, by noting that  $B$  is nonsingular (or nilpotent) if and only if both  $B_K$  and  $B_I$  are nonsingular (or nilpotent). To state the identities, for any combination of symbols  $F, G$  taken from  $\{\text{Mat}, \text{GL}, \text{Nilp}\}$ , we define

$$K_{\zeta, n}^{F \times G} := \{(A, B) \in F_n(\mathbb{F}_q) \times G_n(\mathbb{F}_q) : AB = \zeta BA\}, \quad (\text{IV.2.9})$$

where  $\text{Nilp}_n(\mathbb{F}_q)$  denotes the set of  $n$  by  $n$  nilpotent matrices over  $\mathbb{F}_q$ . Let

$$E_{\zeta}^{F \times G}(x; q) := \sum_{n=0}^{\infty} \frac{|K_{\zeta, n}^{F \times G}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n. \quad (\text{IV.2.10})$$

Then

$$E_{\zeta}^{\text{Mat} \times \text{GL}}(x; q) = E_{\zeta}^{\text{GL} \times \text{GL}}(x; q) E_{\zeta}^{\text{Nilp} \times \text{GL}}(x; q); \quad (\text{IV.2.11})$$

$$E_{\zeta}^{\text{Mat} \times \text{Nilp}}(x; q) = E_{\zeta}^{\text{GL} \times \text{Nilp}}(x; q) E_{\zeta}^{\text{Nilp} \times \text{Nilp}}(x; q). \quad (\text{IV.2.12})$$

Note that Theorem IV.1.2(a) can be restated as

$$E_{\zeta}^{\text{Mat} \times \text{Mat}}(x; q) = E_{\zeta}^{\text{Mat} \times \text{GL}}(x; q) E_{\zeta}^{\text{Mat} \times \text{Nilp}}(x; q). \quad (\text{IV.2.13})$$

### IV.3: Proof of Theorem IV.1.2(b)

Recall that the goal of Theorem IV.1.2(b) is to determine  $|U_{\zeta, n}(\mathbb{F}_q)|$ , namely, to enumerate the pairs of matrices  $(A, B) \in \text{Mat}_n(\mathbb{F}_q) \times \text{Mat}_n(\mathbb{F}_q)$  such that  $AB = \zeta BA$  and  $A$  is nonsingular. To do so, following the approach of Feit and Fine, let  $\beta$  be a similarity class of  $n \times n$  matrices. By a standard orbit-stabilizer argument, for  $B$  in  $\beta$ , the number of nonsingular matrices  $A$  such that  $ABA^{-1} = \zeta B$  is either  $|\text{GL}_n(\mathbb{F}_q)|/|\beta|$  or zero. Moreover, this number is not zero if and only if  $B$  is similar to  $\zeta B$ . We now give a sufficient and necessary condition for it in terms of  $\beta$ .

We recall that each class  $\beta$  corresponds to a unique rational canonical form. It is characterized by an  $n$ -dimensional module  $M_{\beta}$  of the polynomial ring  $\mathbb{F}_q[t]$ . Such a module can be uniquely expressed in the form of

$$M_{\beta} = \frac{\mathbb{F}_q[t]}{(g_1(t))} \oplus \frac{\mathbb{F}_q[t]}{(g_2(t))} \oplus \cdots \oplus \frac{\mathbb{F}_q[t]}{(g_r(t))} \quad (\text{IV.3.1})$$

for monic polynomials  $g_1, \dots, g_r$  such that  $g_i$  divides  $g_{i+1}$  for all  $1 \leq i \leq r-1$ . For a positive integer  $m$ , we say a monic polynomial  $g$  to be in  $\mathcal{P}_m$  if  $g(t) = t^b G(t^m)$  for some nonnegative



integer  $b$  and monic polynomial  $G$ . For example, a polynomial is in  $\mathcal{P}_2$  if it is either even or odd.

**Lemma IV.3.1.** *Let  $B$  be an  $n \times n$  matrix over any field, and let  $\zeta$  be an  $m$ -th root of unity. Then  $B$  is similar to  $\zeta B$  if and only if the polynomials  $g_1, \dots, g_r$  associated to the rational canonical form of  $B$  are in  $\mathcal{P}_m$ .*

*Proof.* We denote the ground field by  $\mathbb{F}$ . An endomorphism  $B$  over a vector space  $V$  determines a module over the polynomial ring  $\mathbb{F}[t]$  by letting  $t \cdot v = Bv$  for  $v \in V$ . We denote this  $\mathbb{F}[t]$ -module by  $(B \curvearrowright V)$ . The isomorphism class of this  $\mathbb{F}[t]$ -module determines the rational canonical form of  $B$ .

Let  $g_1, \dots, g_r$  be the polynomials associated to the rational canonical form of  $B$ . Then

$$(B \curvearrowright V) \cong \frac{\mathbb{F}[t]}{(g_1(t))} \oplus \frac{\mathbb{F}[t]}{(g_2(t))} \oplus \dots \oplus \frac{\mathbb{F}[t]}{(g_r(t))}. \quad (\text{IV.3.2})$$

We now compute  $(\zeta B \curvearrowright V)$ . We have

$$(\zeta B \curvearrowright V) \cong (\zeta t \curvearrowright M_B) \quad (\text{IV.3.3})$$

$$\cong \bigoplus_{i=1}^r \left( \zeta t \curvearrowright \frac{\mathbb{F}[t]}{(g_i(t))} \right) \quad (\text{IV.3.4})$$

$$\cong \bigoplus_{i=1}^r \frac{\mathbb{F}[t]}{(g_i(\zeta^{-1}t))}, \quad (\text{IV.3.5})$$

where the last isomorphism follows from (a): the action of  $\zeta t$  on  $\frac{\mathbb{F}[t]}{(g_i(t))}$  is cyclic, and (b): the polynomial  $x \mapsto g_i(\zeta^{-1}x)$  is a minimal polynomial for  $\zeta t$  acting on  $\frac{\mathbb{F}[t]}{(g_i(t))}$ .

Hence,  $B$  is similar to  $\zeta B$  if and only if

$$\bigoplus_{i=1}^r \frac{\mathbb{F}[t]}{(g_i(t))} \cong \bigoplus_{i=1}^r \frac{\mathbb{F}[t]}{(g_i(\zeta^{-1}t))} \quad (\text{IV.3.6})$$

as  $\mathbb{F}[t]$ -modules. Since  $g_i(t)$  divides  $g_{i+1}(t)$  for all  $i$ , we have that  $g_i(\zeta^{-1}t)$  divides  $g_{i+1}(\zeta^{-1}t)$  as well. By the uniqueness statement about the polynomials associated to the rational canonical form, for each  $i$ , the monic polynomials  $g_i(t)$  and  $\zeta^{\deg g_i} g_i(\zeta^{-1}t)$  must be equal. Write  $g_i(t) = t^d + c_1 t^{d-1} + \dots + c_{d-1} t + c_d$ , then  $\zeta^d g_i(\zeta^{-1}t) = t^d + \zeta c_1 t^{d-1} + \dots + \zeta^{d-1} c_{d-1} t + \zeta^d c_d$ . Since  $\zeta$  is an  $m$ -th root of unity, we observe that  $g_i(t) = \zeta^d g_i(\zeta^{-1}t)$  if and only if  $c_j = 0$  for all  $j$  not divisible by  $m$ . This is equivalent to saying that  $g_i(t)$  is in  $\mathcal{P}_m$ .  $\square$

Let  $\mathcal{S}_{\zeta,n}(\mathbb{F}_q)$  denote the set of similarity classes  $\beta$  of  $n \times n$  matrices over  $\mathbb{F}_q$  such that some (equivalently, every) matrix  $B$  in  $\beta$  is similar to  $\zeta B$ . We have

$$|U_{\zeta,n}(\mathbb{F}_q)| = \sum_{B \in \text{Mat}_n(\mathbb{F}_q)} |\{A \in \text{GL}_n(\mathbb{F}_q) : ABA^{-1} = \zeta B\}| \quad (\text{IV.3.7})$$

$$= \sum_{\beta} \sum_{B \in \beta} |\{A \in \text{GL}_n(\mathbb{F}_q) : ABA^{-1} = \zeta B\}| \quad (\text{IV.3.8})$$

$$= \sum_{\beta \in \mathcal{S}_{\zeta,n}(\mathbb{F}_q)} |\beta| \frac{|\text{GL}_n(\mathbb{F}_q)|}{|\beta|} + \sum_{\beta \notin \mathcal{S}_{\zeta,n}(\mathbb{F}_q)} 0 \quad (\text{IV.3.9})$$

$$= |\text{GL}_n(\mathbb{F}_q)| |\mathcal{S}_{\zeta,n}(\mathbb{F}_q)|. \quad (\text{IV.3.10})$$

We now count  $|\mathcal{S}_{\zeta,n}(\mathbb{F}_q)|$ . By Lemma IV.3.1, a similarity class in  $\mathcal{S}_{\zeta,n}(\mathbb{F}_q)$  is characterized by monic polynomials  $g_1, g_2, \dots, g_r$  in  $\mathcal{P}_m$  such that every polynomial divides the next. Let  $h_i = g_{r+1-i}/g_{r-i}$  for  $1 \leq i \leq t$ , where  $g_0 = 1$ . It is easily checked from the definition of  $\mathcal{P}_m$  that  $g_1, \dots, g_r$  are all in  $\mathcal{P}_m$  if and only if  $h_1, \dots, h_r$  are all in  $\mathcal{P}_m$ . Let  $b_i = \deg h_i$ . The only restriction on the monic  $h_i$  is that  $h_i$  is in  $\mathcal{P}_m$  and that  $\sum_{i=1}^r ib_i = n$ . We observe the important fact that the number of monic polynomials in  $\mathcal{P}_m$  of degree  $b_i$  is  $q^{\lfloor b_i/m \rfloor}$ . Hence to give  $g_1, \dots, g_r$ , we first choose  $(b_i)_{i \geq 1}$  such that  $\sum ib_i = n$ , and then independently choose  $h_i$  in  $\mathcal{P}_m$  of degree  $b_i$ . It follows that

$$|\mathcal{S}_{\zeta,n}(\mathbb{F}_q)| = \sum_{\substack{b_i \geq 0 \\ \sum ib_i = n}} q^{\lfloor b_i/m \rfloor}. \quad (\text{IV.3.11})$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \sum_{n=0}^{\infty} |\mathcal{S}_{\zeta,n}(\mathbb{F}_q)| x^n \quad (\text{IV.3.12})$$

$$= \sum_{n \geq 0} \sum_{\substack{b_i \geq 0 \\ \sum ib_i = n}} q^{\lfloor b_i/m \rfloor} x^n \quad (\text{IV.3.13})$$

$$= \sum_{b_1, b_2, \dots \geq 0} q^{\lfloor b_i/m \rfloor} x^{\sum ib_i} \quad (\text{IV.3.14})$$

$$= \prod_{i=1}^{\infty} \sum_{b=0}^{\infty} q^{\lfloor b/m \rfloor} (x^i)^b. \quad (\text{IV.3.15})$$

By writing  $b = km + l$  with  $0 \leq l < m$ , we get

$$\sum_{b=0}^{\infty} q^{\lfloor b/m \rfloor} x^b = \sum_{l=0}^{m-1} \sum_{k=0}^{\infty} q^k x^{km+l} \quad (\text{IV.3.16})$$

$$= \sum_{l=0}^{m-1} \frac{x^l}{1 - qx^m} \quad (\text{IV.3.17})$$

$$= \frac{1 + x + \cdots + x^{m-1}}{1 - qx^m} \quad (\text{IV.3.18})$$

$$= \frac{1 - x^m}{(1 - x)(1 - qx^m)}. \quad (\text{IV.3.19})$$

Hence, if we define  $G_m(x; q) = \frac{1 - x^m}{(1 - x)(1 - qx^m)}$ , then we have

$$\sum_{n=0}^{\infty} \frac{|U_{\zeta, n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \prod_{i=1}^{\infty} G_m(x^i; q), \quad (\text{IV.3.20})$$

finishing the proof of Theorem [IV.1.2\(b\)](#).

*Remark IV.3.2.* The same argument can compute a similar generating function below, by noting that  $B$  is nonsingular if and only if each polynomial  $g_i(t)$  that appears in the rational canonical form has a nonzero constant term. In the notation of [Remark IV.2.2](#), we have

$$E_{\zeta}^{\text{GL} \times \text{GL}}(x; q) = \prod_{i=1}^{\infty} \frac{1 - x^{im}}{1 - x^{im}q}, \quad (\text{IV.3.21})$$

where  $m = \text{ord}(\zeta)$ .

Similarly, if we instead notice that  $B$  is nilpotent if and only if each  $g_i(t)$  is a power of  $t$ , we get

$$E_{\zeta}^{\text{GL} \times \text{Nilp}}(x; q) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}. \quad (\text{IV.3.22})$$

We notice that the above two formulas, together with [Theorem IV.1.2\(b\)](#), verify [\(IV.2.11\)](#) explicitly. We also observe that  $E_{\zeta}^{\text{GL} \times \text{GL}}(x; q)$  is a power series in  $x^m$ . In particular, this implies that if  $AB = \zeta BA$  and  $A, B$  are both nonsingular, then the size  $n$  of the matrices  $A, B$  must be a multiple of the order of  $\zeta$ .



is of the following form:

$$\left[ \begin{array}{c|cc|cc|cc|c|c|c|c|c}
B_{1,1}^1 & & B_{1,2}^1 & & & B_{1,3}^1 & & & & B_{1,4}^1 & \cdots \\
\hline
B_{2,1}^1 & B_{2,2}^1 & B_{2,2}^2 & & B_{2,3}^1 & B_{2,3}^2 & & & & B_{2,4}^1 & B_{2,4}^2 & \cdots \\
& & \zeta B_{2,2}^1 & & & \zeta B_{2,3}^1 & & & & & \zeta B_{2,4}^1 & \cdots \\
\hline
B_{3,1}^1 & B_{3,2}^1 & B_{3,2}^2 & B_{3,3}^1 & B_{3,3}^2 & B_{3,3}^3 & & B_{3,4}^1 & B_{3,4}^2 & B_{3,4}^3 & \cdots \\
& & \zeta B_{3,2}^1 & & \zeta B_{3,3}^1 & \zeta B_{3,3}^2 & & & \zeta B_{3,4}^1 & \zeta B_{3,4}^2 & \cdots \\
& & & & & \zeta^2 B_{3,3}^1 & & & & \zeta^2 B_{3,4}^1 & \cdots \\
\hline
B_{4,1}^1 & B_{4,2}^1 & B_{4,2}^2 & B_{4,3}^1 & B_{4,3}^2 & B_{4,3}^3 & B_{4,4}^1 & B_{4,4}^2 & B_{4,4}^3 & B_{4,4}^4 & \cdots \\
& & \zeta B_{4,2}^1 & & \zeta B_{4,3}^1 & \zeta B_{4,3}^2 & & \zeta B_{4,4}^1 & \zeta B_{4,4}^2 & \zeta B_{4,4}^3 & \cdots \\
& & & & & \zeta^2 B_{4,3}^1 & & & \zeta^2 B_{4,4}^1 & \zeta^2 B_{4,4}^2 & \cdots \\
& & & & & & & & & \zeta^3 B_{4,4}^1 & \cdots \\
\hline
\vdots & & \vdots & & \vdots & & & \vdots & & \vdots & \ddots
\end{array} \right], \quad (\text{IV.4.4})$$

where each  $B_{i,j}^k$  is an arbitrary  $a_i$  by  $a_j$  matrix, chosen independently. We note that the count  $|\{B \in \text{Mat}_n(\mathbb{F}_q) : A_\pi B = \zeta B A_\pi\}|$  does not depend on  $\zeta$ . Hence,

$$|N_{\zeta,n}(\mathbb{F}_q)| = |N_{1,n}(\mathbb{F}_q)|. \quad (\text{IV.4.5})$$

It is known in [FF60, Equation (6)] that

$$|N_{1,n}(\mathbb{F}_q)| = |\text{GL}_n(\mathbb{F}_q)| \sum_{\pi \vdash n} \frac{1}{f(a_1)f(a_2)\dots}, \quad (\text{IV.4.6})$$

where  $f(a) := (1 - q^{-1})(1 - q^{-2}) \dots (1 - q^{-a})$ .

Hence,

$$\sum_{n=0}^{\infty} \frac{|N_{\zeta,n}(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} x^n = \sum_{n=0}^{\infty} \sum_{\pi \vdash n} \frac{1}{f(a_1)f(a_2)\dots} x^n \quad (\text{IV.4.7})$$

$$= \sum_{a_1, a_2, \dots \geq 0} \frac{1}{f(a_1)f(a_2)\dots} x^{\sum ia_i} \quad (\text{IV.4.8})$$

$$= \prod_{i=1}^{\infty} \sum_{a=0}^{\infty} \frac{1}{f(a)} (x^i)^a \quad (\text{IV.4.9})$$

$$= \prod_{i=1}^{\infty} H(x^i; q), \quad (\text{IV.4.10})$$

where

$$H(x; q) := \sum_{a=0}^{\infty} \frac{1}{f(a)} x^a = \frac{1}{(1-x)(1-xq^{-1})(1-xq^{-2}) \dots} \quad (\text{IV.4.11})$$

by a classical identity due to Euler. This concludes the proof of Theorem IV.1.2(c), and hence proves Theorem IV.1.2 and Theorem IV.1.1.

*Remark IV.4.1.* Combining Theorem IV.1.2(c), formula (IV.3.22) and the decomposition formula (IV.2.12), we get (in the notation of Remark IV.2.2)

$$E_{\zeta}^{\text{Nilp} \times \text{Nilp}}(x; q) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i q^{-1})(1-x^i q^{-2}) \dots}. \quad (\text{IV.4.12})$$

At this point, we have computed  $E_{\zeta}^{\text{F} \times \text{G}}(x; q)$  for all combinations of  $\text{F}, \text{G} \in \{\text{Mat}, \text{GL}, \text{Nilp}\}$ . We notice that  $E_{\zeta}^{\text{F} \times \text{G}}(x; q)$  does not depend on  $\zeta$  whenever  $\text{F}$  or  $\text{G}$  is  $\text{Nilp}$ . This should not be surprising in light of the argument of Theorem IV.1.2(c).

## IV.5: Discussions

We note from the work of Bryan and Morrison [BM15, §3.1] that  $|U_{1,n}(\mathbb{F}_q)|$  and  $|N_{1,n}(\mathbb{F}_q)|$  “determine” each other. The key ingredient is that either of the quantities above is the point count of the variety of modules over the “commutative” plane  $\text{Spec } \mathbb{F}_q[X, Y]$  supported on a certain subset of closed points. A module is determined by its localizations at closed points in its support, so both  $|U_{1,n}(\mathbb{F}_q)|$  and  $|N_{1,n}(\mathbb{F}_q)|$  are determined by the point count of the variety of modules supported at a point. Since the commutative plane “looks the same everywhere” locally in light of the Cohen structure theorem (the complete localization of  $\mathbb{F}_q[X, Y]$  at any closed point is isomorphic to  $\mathbb{F}[[X, Y]]$  for some field extension  $\mathbb{F}$  of  $\mathbb{F}_q$ ), we can reverse the process, so that either of  $|U_{1,n}(\mathbb{F}_q)|$  and  $|N_{1,n}(\mathbb{F}_q)|$  determines the point count of the variety of modules supported at a point, and hence determines each other.

However, for  $\zeta \neq 1$ , Theorem IV.1.2 shows that  $|N_{\zeta,n}(\mathbb{F}_q)|$  does not depend on  $\zeta$  while  $|U_{\zeta,n}(\mathbb{F}_q)|$  does. Is it still possible to recover  $|U_{\zeta,n}(\mathbb{F}_q)|$  from  $|N_{\zeta,n}(\mathbb{F}_q)|$  together with the geometry of the quantum plane  $XY = \zeta YX$  (which will depend on  $\zeta$ )?

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# CHAPTER V

## Rationality for the Betti Numbers of the Unordered Configuration Spaces of a Punctured Torus

The content of this chapter is a joint work with Gilyoung Cheong, available at [\[CH22\]](#).

### Abstract

Given an elliptic curve  $E$  defined over  $\mathbb{C}$ , let  $E^\times$  be an open subset of  $E$  obtained by removing a point. In this paper, we show that the  $i$ -th Betti number of the unordered configuration space  $\text{Conf}^n(E^\times)$  of  $n$  points on  $E^\times$  appears as a coefficient of an explicit rational function in two variables. We also compute its Hodge numbers as coefficients of another explicit rational function in four variables. Our result is interesting because these rational functions resemble the generating function of the  $\mathbb{F}_q$ -point counts of  $\text{Conf}^n(E^\times)$ , which can be obtained from the zeta function of  $E$  over any fixed finite field  $\mathbb{F}_q$ . We show that the mixed Hodge structure of the  $i$ -th singular cohomology group  $H^i(\text{Conf}^n(E^\times))$  with complex coefficients is pure of weight  $w(i)$ , an explicit integer we provide in this paper. This purity statement implies our main result about the Betti numbers and the Hodge numbers. Our proof uses Totaro's spectral sequence computation that describes the weight filtration of the mixed Hodge structure on  $H^i(\text{Conf}^n(E^\times))$ .

### V.1: Introduction

#### V.1.1: Motivation and main results

In number theory, it is classically known that the probability that a random positive integer is square-free is equal to  $1/\zeta_{\text{Spec}(\mathbb{Z})}(2)$ , where  $\zeta_{\text{Spec}(\mathbb{Z})}(s)$  is the Riemann zeta function. More specifically, we have

$$\lim_{n \rightarrow \infty} \text{Prob} \left( m \text{ is square-free} \right) = \frac{1}{\zeta_{\text{Spec}(\mathbb{Z})}(2)}. \quad (\text{V.1.1})$$

The analogue for  $\mathbb{F}_q[x]$  in place of  $\mathbb{Z}$ , where  $\mathbb{F}_q$  is a finite field, is even better: we have

$$\text{Prob}_{\substack{f \in \mathbb{F}_q[x]: \\ f \text{ monic and} \\ \deg(f)=n}} (f \text{ is square-free}) = 1 - q^{-1} = \frac{1}{\zeta_{\mathbb{A}_{\mathbb{F}_q}^1}(2)}, \quad (\text{V.1.2})$$

for any fixed degree  $n \geq 2$ , where  $\zeta_{\mathbb{A}_{\mathbb{F}_q}^1}(s)$  is the zeta function of the affine line  $\mathbb{A}_{\mathbb{F}_q}^1 = \text{Spec}(\mathbb{F}_q[x])$  over  $\mathbb{F}_q$ . In [CEF14], Church, Ellenberg, and Farb explain how counting such polynomials is related to the topology of the unordered configuration space  $\text{Conf}^n(\mathbb{C})$  of the complex plane  $\mathbb{C} = \mathbb{R}^2$ , or equivalently, the set of  $\mathbb{C}$ -points on the affine line over  $\mathbb{C}$ . After realizing the set of monic square-free polynomials in  $\mathbb{F}_q[x]$  as the set  $\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)$  of the  $\mathbb{F}_q$ -points on the unordered configuration space of the affine line, they show that

$$|\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)| = \sum_{i=0}^{\infty} (-1)^i q^{n-i} h^i(\text{Conf}^n(\mathbb{C})), \quad (\text{V.1.3})$$

where  $h^i$  means the  $i$ -th Betti number, using the  $l$ -adic cohomology theory along with the result about how the geometric Frobenius acts on the  $i$ -th  $l$ -adic cohomology group of an arbitrary affine hyperplane complement over  $\mathbb{F}_q$ , independently known by Lehrer [Leh92a] and Kim [Kim94]. Note that

$$\sum_{n=0}^{\infty} |\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)| t^n = \sum_{\substack{f \in \mathbb{F}_q[x]: \\ f \text{ monic and} \\ \text{square-free}}} t^{\deg(f)} = \prod_{P \in |\mathbb{A}_{\mathbb{F}_q}^1|} (1 + t^{\deg(P)}) \quad (\text{V.1.4})$$

$$= \prod_{P \in |\mathbb{A}_{\mathbb{F}_q}^1|} \frac{1 - t^{2 \deg(P)}}{1 - t^{\deg(P)}} = \frac{Z_{\mathbb{A}_{\mathbb{F}_q}^1}(t)}{Z_{\mathbb{A}_{\mathbb{F}_q}^1}(t^2)} = \frac{1 - qt^2}{1 - qt}, \quad (\text{V.1.5})$$

where  $|\mathbb{A}_{\mathbb{F}_q}^1|$  is the set of monic irreducible polynomials in  $\mathbb{F}_q[x]$  and  $Z_{\mathbb{A}_{\mathbb{F}_q}^1}(t)$  is the zeta series of  $\mathbb{A}_{\mathbb{F}_q}^1$ , meaning  $Z_{\mathbb{A}_{\mathbb{F}_q}^1}(q^{-s}) = \zeta_{\mathbb{A}_{\mathbb{F}_q}^1}(s)$ . Hence, we have

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i q^{n-i} h^i(\text{Conf}^n(\mathbb{C})) t^n = \frac{1 - qt^2}{1 - qt} = 1 + qt + (q^2 - q)t^2 + (q^3 - q^2)t^3 + \dots \quad (\text{V.1.6})$$

Since the above identity holds for all prime powers  $q$ , we have

$$h^i(\text{Conf}^n(\mathbb{C})) = \begin{cases} 1 & \text{if } n = 0, 1 \text{ and } i = 0, \\ 1 & \text{if } n \geq 2 \text{ and } i = 0, 1, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{V.1.7})$$



which recovers a result of Arnol'd in [Arn69].

In this paper, we seek a genus 1 analogue of this story by replacing  $\mathbb{A}^1$ , or  $\mathbb{P}^1$  minus a (degree 1) point, by an elliptic curve  $E$  minus a point, which can be defined by any equation of the form

$$y^2 = f(x), \quad (\text{V.1.8})$$

where  $f(x)$  is a square-free polynomial of degree 3 over the ambient field. We denote this punctured elliptic curve as  $E^\times$ . Over  $\mathbb{F}_q$ , a theorem of Weil gives an explicit form of the zeta series

$$Z_{E^\times}(t) = \frac{(1 - \alpha t)(1 - \bar{\alpha} t)}{1 - qt}, \quad (\text{V.1.9})$$

where  $\alpha$  is an algebraic integer with the complex norm  $q^{1/2}$ , and  $\bar{\alpha}$  is its complex conjugation. This implies that

$$\sum_{n=0}^{\infty} |\text{Conf}^n(E^\times)(\mathbb{F}_q)| t^n = \frac{Z_{E^\times}(t)}{Z_{E^\times}(t^2)} = \frac{(1 - \alpha t)(1 - \bar{\alpha} t)(1 - qt^2)}{(1 - \alpha t^2)(1 - \bar{\alpha} t^2)(1 - qt)}, \quad (\text{V.1.10})$$

using a similar computation for the  $\mathbb{A}^1$  case. Unfortunately, the proof presented in [CEF14] does not generalize to connect this arithmetic result to topology. As our main result, we obtain a topological analogue of the above arithmetic computation with a different proof:

**Theorem V.1.1.** *Let  $E^\times$  be an open subset of an elliptic curve over  $\mathbb{C}$  obtained by removing a point, with respect to the analytic topology. We have*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i h^i(\text{Conf}^n(E^\times)) u^{2n-w(i)} t^n = \frac{(1 - ut)^2(1 - u^2 t^2)}{(1 - ut^2)^2(1 - u^2 t)}, \quad (\text{V.1.11})$$

and

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \sum_{p,q \geq 0} h^{n-p,n-q}(H^i(\text{Conf}^n(E^\times))) x^p y^q u^{2n-w(i)} t^n = \frac{(1 - xut)(1 - yut)(1 - xyu^2 t^2)}{(1 - xut^2)(1 - yut^2)(1 - xyu^2 t)}, \quad (\text{V.1.12})$$

where

$$w(i) := \begin{cases} 3i/2 & \text{if } i \text{ is even and} \\ (3i - 1)/2 & \text{if } i \text{ is odd.} \end{cases} \quad (\text{V.1.13})$$

and  $h^{p,q}(H^i(\text{Conf}^n(E^\times)))$  denote the Hodge numbers of the  $i$ -th cohomology group<sup>1</sup> of

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<sup>1</sup>From now on, every cohomology of a complex variety we deal with is assumed to be its singular cohomology with complex coefficients with respect to the analytic topology. Furthermore, every variety is assumed to be defined over complex numbers, unless mentioned otherwise, and we will identify a variety with its set of complex points.

$\text{Conf}^n(E^\times)$ . In particular, since  $w : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  is injective, the rational function computes all the Hodge numbers.

*Remark V.1.2.* We note that  $\text{Conf}^n(E^\times)$  is smooth but not projective (nor proper), so to discuss its Hodge theory, we need Deligne's mixed Hodge structure, introduced in [Del71]. Even the first statement of Theorem V.1.1 about the Betti numbers of  $\text{Conf}^n(E^\times)$  will be deduced from the following, a vanishing statement about the Hodge numbers, which is our major contribution. Thus, the computation of the Betti numbers can be regarded as a concrete application of Deligne's theory.

**Theorem V.1.3.** *For any  $i \in \mathbb{Z}_{\geq 0}$ , the mixed Hodge structure of the  $i$ -th cohomology group  $H^i(\text{Conf}^n(E^\times))$  of  $\text{Conf}^n(E^\times)$  is pure of weight  $w(i)$ , the number mentioned in Theorem V.1.1. In other words, we have  $h^{p,q}(H^i(\text{Conf}^n(E^\times))) = 0$  unless  $p + q = w(i)$ .*

*Remark V.1.4.* There are direct genus 0 analogues of Theorem V.1.1 and Theorem V.1.3 with  $\mathbb{P}^1(\mathbb{C})$  replacing  $E$  and thus with  $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$  replacing  $E^\times$ . It follows from [Kim94, Theorem 1] that the  $i$ -th cohomology group of  $\text{Conf}^n(\mathbb{C})$  has a pure Hodge structure of weight  $2i$ , meaning that  $h^{p,q}(H^i(\text{Conf}^n(\mathbb{C}))) = 0$  unless  $p + q = 2i$ . This is the genus 0 analogue of Theorem V.1.3, which is the genus 1 case. Just as our paper deduces Theorem V.1.1 from Theorem V.1.3, this genus 0 purity result implies that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i h^i(\text{Conf}^n(\mathbb{C})) u^{2n-2i} t^n = \frac{1 - u^2 t^2}{1 - u^2 t} \quad (\text{V.1.14})$$

$$= 1 + u^2 t + (u^4 - u^2) t^2 + (u^6 - u^4) t^3 + \dots, \quad (\text{V.1.15})$$

which is analogous to the first part<sup>2</sup> of Theorem V.1.1. One may note that replacing  $u^2$  with  $q$  recovers the generating function for  $|\text{Conf}^n(\mathbb{A}^1)(\mathbb{F}_q)|$ , introduced in the beginning.

For the case of  $\text{Conf}^n(\mathbb{C})$ , in contrast to our case of  $\text{Conf}^n(E^\times)$ , as seen in [Kim94], the  $i$ -th cohomology group  $H^i(F(\mathbb{C}, n))$  of the ordered configuration space  $F(\mathbb{C}, n)$  of  $n$  points on  $\mathbb{C}$  has a pure Hodge structure of weight  $2i$ , which implies that  $H^i(\text{Conf}^n(\mathbb{C}))$  also has a pure Hodge structure of weight  $2i$ . However, it turns out that  $H^i(F(E^\times, n))$ , replacing  $\mathbb{C}$  with  $E^\times$ , does not have a pure Hodge structure in general (e.g., Example 4.2 of [Bib16]), so our approach to Theorem V.1.3 has to be significantly different. We use the fact that the  $S_n$ -invariance of the Leray spectral sequence  $(E_{p,q}^r)_{p,q,r}$  of the inclusion

$$F(E^\times, n) \hookrightarrow (E^\times)^n \quad (\text{V.1.16})$$

---

<sup>2</sup>One may also deduce the analogue of the second part of Theorem V.1.1, but we omit this discussion.

converges to the cohomology group we want to compute and degenerates at the  $E_3$  page, which turns out to remember the weight filtration of the cohomology. (Recall that for showing the purity, understanding the weight filtration is enough.) Moreover, the  $E_2$  page of this spectral sequence can be explicitly described using a work of Totaro in [Tot96], so we use this description, which we review in Section V.3, to approach our problem in an explicit manner.

### V.1.2: Related works and future directions

The literature of cohomology of the (either ordered or unordered) configuration spaces of a manifold is extensive, so we refer to only a few of them directly related to our work. The Betti numbers  $h^i(\text{Conf}^n(E^\times))$  were first computed by Bödigheimer and Cohen [BC88]. (See [DCK17, Proposition 3.5] for more explicit computations.) They were also studied by Napolitano [Nap03, p.489, Table 3]. The new features of Theorem V.1.1 are the computation of Hodge numbers  $h^{p,q}(H^i(\text{Conf}^n(E^\times)))$  and the explicit rational generating functions for either Betti or Hodge numbers. A natural follow-up question is to ask whether we can prove analogues of Theorem V.1.1 in other examples. For instance, we may consider  $\text{Conf}^n(E \setminus \{p_1, p_2\})$ , where  $p_1$  and  $p_2$  are distinct points of  $E$ . Then applying a theorem of the second author in [Hua20a], we can obtain an analogue of Theorem V.1.1 by relating  $\text{Conf}^n(E^\times)$  and  $\text{Conf}^n(E \setminus \{p_1, p_2\})$  although it turns out that for some  $i$ , the  $i$ -th cohomology group  $H^i(\text{Conf}^n(E \setminus \{p_1, p_2\}))$  does not have purity as in Theorem V.1.3. In particular, the purity is a sufficient but not a necessary condition to have a rational generating function for Betti or Hodge numbers. Similarly, we can obtain an analogue of Theorem V.1.1 by replacing  $E^\times$  with  $E$  minus  $r$  points for any  $r \in \mathbb{Z}_{\geq 1}$ . However, the result in [Hua20a] does not work for relating  $\text{Conf}^n(E)$  and  $\text{Conf}^n(E^\times)$ . We also note that our approach uses the fact that the second cohomology of  $E^\times$  is trivial, so for the case of  $\text{Conf}^n(E)$ , the computations get more complicated. Fortunately, according to Pagaria in [Pag], one can use a structure of  $\mathfrak{sl}(2)$ -representation on the graded quotients of  $H^\bullet(\text{Conf}^n(E))$  with respect to the weight filtration to compute the Hodge numbers in terms of a power series. (See the remark following [Pag, Theorem 4.11].) However, it is still unclear how to obtain an analogue to Theorem V.1.1 as we cannot tell such a power series is rational. Pagaria's method also deals with  $H^\bullet(\text{Conf}^n(X))$  for a smooth projective curve  $X$  of any genus  $g$ , for which he uses  $\mathfrak{sl}(2g)$  instead of  $\mathfrak{sl}(2)$ . Our method in this paper also applies for  $H^\bullet(\text{Conf}^n(X^\times))$ , where  $X^\times$  is  $X$  minus a point, but we do not get the analogue of Theorem V.1.3, so we are unable to obtain the analogue of Theorem V.1.1 yet. Nevertheless, we conjecture that the rationality of generating functions as in Theorem V.1.1 holds for a large family of varieties in place of  $E^\times$ , even without purity. This opens up many problems that can be potentially resolved by means

of algebraic geometry, combinatorics, topology, or other relevant areas. The problem also does not restrict to configuration spaces; for example, Farb, Wolfson, and Wood [FWW19, Theorem 3.1, Statement 3] discuss a generalization of configuration spaces with which they generalize Totaro’s work in [Tot96].

### V.1.3: Organization of the paper

In Section V.2, we explain why the purity statement given by Theorem V.1.3 implies Theorem V.1.1. We review Totaro’s work in Section V.3, which is a key tool for our proof of Theorem V.1.3. In Section V.4, we prove Theorem V.1.3.

## V.2: Theorem V.1.3 implies Theorem V.1.1

In this section, we explain how Theorem V.1.3 implies Theorem V.1.1. Our work depends on the mixed Hodge structure on the cohomology of algebraic varieties, and we use some of their known properties.<sup>3</sup> Consider the polynomial

$$\chi(X, u) := \sum_{i,j=0}^{\infty} (-1)^i \dim_{\mathbb{C}}(\mathrm{Gr}_W^j(H_c^i(X))) u^j, \quad (\text{V.2.1})$$

where  $H_c^i(X)$  means the  $i$ -th compactly supported (singular) cohomology group of a variety  $X$  with complex coefficients and  $\mathrm{Gr}_W^j(H_c^i(X))$  means its  $j$ -th graded quotient with respect to the weight filtration on  $H_c^i(X)$ .

**Example V.2.1.** If  $X$  is smooth and projective, then the Hodge structure on  $H_c^i(X) = H^i(X)$  is pure of weight  $i$ .<sup>4</sup> In other words, we have

$$\mathrm{Gr}_W^j(H_c^i(X)) = \begin{cases} H_c^i(X) & \text{if } j = i \text{ and} \\ 0 & \text{if } j \neq i. \end{cases} \quad (\text{V.2.2})$$

Thus, in this case, we have

$$\chi(X, u) = \sum_{i=0}^{\infty} (-1)^i h_c^i(X) u^i = \sum_{i=0}^{\infty} (-1)^i h^i(X) u^i, \quad (\text{V.2.3})$$

where  $h_c^i(X) := \dim_{\mathbb{C}}(H_c^i(X))$ . This implies that  $\chi(X, 1) = \chi(X)$ , the Euler characteristic of  $X$ . For our purpose, we will only deal with varieties that are smooth, but not necessarily

<sup>3</sup>These properties are reviewed, for example, in a paper by Danilov and Khovanski [DK86, Section 1].

<sup>4</sup>In this case, we simply say that “ $H_c^i(X)$  is pure of weight  $i$ ,” and similarly for other weights for any pure Hodge structure.

projective nor proper.

For any closed subvariety  $Z$  of any variety  $X$ , it is known (e.g., from [FM94, p. 198]) that

$$\chi(X, u) = \chi(Z, u) + \chi(X \setminus Z, u). \quad (\text{V.2.4})$$

Since the weight filtration on  $H_c^i(X)$  is compatible with the Künneth formula, we also get

$$\chi(X \times Y, u) = \chi(X, u)\chi(Y, u) \quad (\text{V.2.5})$$

for any two varieties  $X$  and  $Y$ . We call these two properties the **motivic properties** of  $\chi(-, u)$ . Given a quasi-projective variety  $X$ , the  $n$ -th symmetric power  $\text{Sym}^n(X) = X^n/S_n$  exists as a variety for every  $n \in \mathbb{Z}_{\geq 0}$ , whose set of  $\mathbb{C}$ -points corresponds to the quotient topological space. We define two power series

$$Z(X, u, t) := \sum_{n=0}^{\infty} \chi(\text{Sym}^n(X), u)t^n \quad (\text{V.2.6})$$

and

$$K(X, u, t) := \sum_{n=0}^{\infty} \chi(\text{Conf}^n(X), u)t^n. \quad (\text{V.2.7})$$

Due to the motivic properties of  $\chi(-, u)$ , it follows from Proposition 5.9 of [VW15] that

$$K(X, u, t) = \frac{Z(X, u, t)}{Z(X, u, t^2)}. \quad (\text{V.2.8})$$

### V.2.1: Theorem V.1.3 implies the first part of Theorem V.1.1

Consider a smooth quasi-projective variety  $X$ . If  $H^i(X)$  is pure of weight  $w(i)$ , possibly different from  $i$ , then  $H_c^{2n-i}(X)$  is pure of weight  $2n - w(i)$ , so

$$\chi(X, u) = \sum_{i=0}^{\infty} (-1)^i h_c^{2n-i}(X) u^{2n-w(i)} \quad (\text{V.2.9})$$

$$= \sum_{i=0}^{\infty} (-1)^i h^i(X) u^{2n-w(i)}, \quad (\text{V.2.10})$$

where  $n$  is the (complex) dimension of  $X$ . Hence, if we assume Theorem V.1.3, the first part of Theorem V.1.1 merely says

$$K(E^\times, u, t) = \frac{(1-ut)^2(1-u^2t^2)}{(1-ut^2)^2(1-u^2t)}. \quad (\text{V.2.11})$$

Using a long exact sequence and Poincaré duality, we observe that  $H^i(E^\times) \simeq H^i(E)$  for  $i = 0, 1$  and that these isomorphisms preserve the mixed Hodge structures.<sup>5</sup> Since  $H^i(E^\times) = 0$  for  $i \geq 2$ , we see that each  $H^i(E^\times)$  is pure of weight  $i$  so that  $H^i((E^\times)^n)$  is pure of weight  $i$ , using the fact that the Künneth isomorphism preserves the mixed Hodge structures. Thus, it follows that  $H_c^{2n-i}((E^\times)^n)$  is pure of weight  $2n-i$  so that  $H_c^i((E^\times)^n)$  is pure of weight  $i$ . This implies that  $H_c^i(\text{Sym}^n(E^\times))$  is pure of weight  $i$  because

$$H_c^i(\text{Sym}^n(E^\times)) \simeq H_c^i((E^\times)^n)^{S_n} \hookrightarrow H_c^i((E^\times)^n) \quad (\text{V.2.12})$$

induced by the quotient map  $(E^\times)^n \rightarrow (E^\times)^n/S_n = \text{Sym}^n(E^\times)$ , which is a finite map, and all the maps above are strictly compatible with the mixed Hodge structures. Thus, we have

$$\chi(\text{Sym}^n(E^\times), u) = \sum_{i=0}^{\infty} (-1)^i h_c^i(\text{Sym}^n(E^\times)) u^i, \quad (\text{V.2.13})$$

and by a formula due to Macdonald (originally from [Mac62] but we use the version given in Cheah's thesis [Che94, p.116] for the compactly supported cohomology), we have

$$Z(E^\times, u, t) = \sum_{n=0}^{\infty} \chi(\text{Sym}^n(E^\times), u) t^n \quad (\text{V.2.14})$$

$$= \frac{(1-ut)^{h_c^1(E^\times)}}{(1-t)^{h_c^0(E^\times)}(1-u^2t)^{h_c^2(E^\times)}} \quad (\text{V.2.15})$$

$$= \frac{(1-ut)^{h^1(E^\times)}}{(1-t)^{h^2(E^\times)}(1-u^2t)^{h^0(E^\times)}} \quad (\text{V.2.16})$$

$$= \frac{(1-ut)^2}{1-u^2t}. \quad (\text{V.2.17})$$

Therefore, we have

$$K(E^\times, u, t) = \frac{Z(E^\times, u, t)}{Z(E^\times, u, t^2)} = \frac{(1-ut)^2(1-u^2t^2)}{(1-u^2t)(1-ut^2)^2}, \quad (\text{V.2.18})$$

so this proves that Theorem V.1.3 implies the first part of Theorem V.1.1.

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<sup>5</sup>See [DK86, p.282].

**V.2.2: Theorem V.1.3 implies the second part of Theorem V.1.1**

For any variety  $X$ , consider the mixed Hodge polynomial<sup>6</sup>

$$\mathbf{H}_c(X, x, y, u) := \sum_{p, q, i \geq 0} h^{p, q}(H_c^i(X)) x^p y^q (-u)^i. \quad (\text{V.2.19})$$

This is a generating function for the Hodge numbers

$$h^{p, q}(H_c^i(X)) := \dim_{\mathbb{C}}(\text{Gr}_p^F \text{Gr}_W^{p+q}(H_c^i(X))), \quad (\text{V.2.20})$$

where  $\text{Gr}_p^F$  denotes the  $p$ -th graded quotient of the Hodge filtration. Consider the following generating function:

$$\mathbf{Z}(X, x, y, u, t) := \sum_{n=0}^{\infty} \mathbf{H}_c(\text{Sym}^n(X), x, y, u) t^n, \quad (\text{V.2.21})$$

defined for any quasi-projective variety  $X$ . Using the fact that the Hodge numbers of  $E$  are symmetric (i.e.,  $h^{p, q}(E) = h^{q, p}(E)$ ), we can compute each Hodge number  $h^{p, q}(E)$ . From this and a long exact sequence, we can compute the Hodge numbers of  $E^\times$  so that

$$\mathbf{H}_c(E^\times, x, y, u) = -(x + y)u + xyu^2. \quad (\text{V.2.22})$$

Hence, using a formula due to Cheah [Che94, p.116], we obtain that the generating function for  $X = E^\times$  is given by

$$\mathbf{Z}(E^\times, x, y, u, t) = \frac{(1 - xut)(1 - yut)}{(1 - xyu^2t)}. \quad (\text{V.2.23})$$

Recall that  $\mathbf{H}_c(-, x, y, 1)$  satisfies the motivic properties (e.g., from [DK86, Propositions 1.6 and 1.8]). By [VW15, Proposition 5.9], we have

$$\sum_{n=0}^{\infty} \chi(\text{Conf}^n(E^\times), x, y, 1) t^n = \frac{\mathbf{Z}(E^\times, x, y, 1, t)}{\mathbf{Z}(E^\times, x, y, 1, t^2)} \quad (\text{V.2.24})$$

$$= \frac{(1 - xt)(1 - yt)(1 - xyt^2)}{(1 - xyt)(1 - xt^2)(1 - yt^2)}. \quad (\text{V.2.25})$$

---

<sup>6</sup>In the literature, the ‘‘mixed Hodge polynomial’’ often means the specialization  $u = 1$  of the polynomial  $\mathbf{H}_c(X, x, y, u)$  we use here, which may not remember all the Hodge numbers. The polynomial  $\mathbf{H}_c(X, x, y, u)$  we use here does remember all the Hodge numbers.

This implies that

$$\sum_{p,q,i,n \geq 0} (-1)^i h^{p,q}(H_c^{2n-i}(\text{Conf}^n(E^\times))) x^p y^q t^n = \sum_{p,q,i,n \geq 0} (-1)^i h^{p,q}(H_c^i(\text{Conf}^n(E^\times))) x^p y^q t^n \quad (\text{V.2.26})$$

$$= \frac{(1-xt)(1-yt)(1-xyt^2)}{(1-xyt)(1-xt^2)(1-yt^2)}. \quad (\text{V.2.27})$$

We now use Theorem V.1.3, which tells us that

$$h^{p,q}(H_c^{2n-i}(\text{Conf}^n(E^\times))) = h^{n-p,n-q}(H^i(\text{Conf}^n(E^\times))) = 0 \quad (\text{V.2.28})$$

unless  $p + q = 2n - w(i)$ , so

$$\sum_{\substack{p,q,i,n \geq 0, \\ p+q=2n-w(i)}} (-1)^i h^{n-p,n-q}(H^i(\text{Conf}^n(E^\times))) x^p y^q t^n = \frac{(1-xt)(1-yt)(1-xyt^2)}{(1-xyt)(1-xt^2)(1-yt^2)}. \quad (\text{V.2.29})$$

Thus, we may replace  $x$  and  $y$  by  $xu$  and  $yu$  respectively to get

$$\sum_{\substack{p,q,i,n \geq 0, \\ p+q=2n-w(i)}} (-1)^i h^{n-p,n-q}(H^i(\text{Conf}^n(E^\times))) x^p y^q u^{2n-w(i)} t^n = \frac{(1-xut)(1-yut)(1-xyu^2t^2)}{(1-xyut)(1-xut^2)(1-yut^2)}, \quad (\text{V.2.30})$$

and the above identity holds without specifying the conditions  $p + q = 2n - w(i)$ , because all the coefficients on the left-hand side violating such conditions are equal to 0 by Theorem V.1.3. Thus, we obtain the second part of Theorem V.1.1 from Theorem V.1.3.

## V.3: Totaro's description of a Leray spectral sequence

### V.3.1: Recalling Poincaré duality

We recall the following explicit form of Poincaré duality (e.g., from [Ful95, Theorem 24.18]). That is, given any oriented real manifold  $M$  of dimension  $m$ , Poincaré duality is given by the  $\mathbb{R}$ -linear isomorphism

$$H^i(M, \mathbb{R}) \simeq H_c^{m-i}(M, \mathbb{R})^\vee \quad (\text{V.3.1})$$

given by

$$[\omega] \mapsto \left( [\mu] \mapsto \int_M \omega \wedge \mu \right), \quad (\text{V.3.2})$$



where  $H^i(M, \mathbb{R})$  denotes the  $i$ -th de Rham cohomology of  $M$  with real coefficients and similarly for the compactly supported de Rham cohomology  $H_c^i(M, \mathbb{R})$ . We wrote  $V^\vee := \text{Hom}_k(V, k)$  for the dual vector space of a vector space  $V$  over a field  $k$ . When  $M$  is a complex manifold of (complex) dimension  $n$ , then it is an oriented real manifold of dimension  $2n$ , so applying  $(-)\otimes_{\mathbb{R}}\mathbb{C}$  to the above isomorphism gives

$$H^i(M) \simeq H_c^{2n-i}(M)^\vee. \quad (\text{V.3.3})$$

### V.3.2: Totaro's work

Our approach to attack Theorem V.1.3 is to use the Leray spectral sequence of the inclusion  $F(X, n) \hookrightarrow X^n$  for a variety  $X$ , where

$$F(X, n) := \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ whenever } i \neq j\} \quad (\text{V.3.4})$$

so that  $\text{Conf}^n(X) = F(X, n)/S_n$ . We are only interested in the case  $X = E^\times$ . The  $E_2$  page of this spectral sequence is described by Totaro in [Tot96], and we closely follow [Tot96, Theorem 3]. For our specific  $X = E^\times$ , the differentials  $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q+1-r}$  are only possibly nonzero at the page  $r = 2$ , so  $E_\infty^{p,q} = E_3^{p,q}$ . Most importantly, Totaro shows that

$$E_3^{p,q} = \text{Gr}_{p+2q}^W(H^{p+q}(F(X, n))). \quad (\text{V.3.5})$$

We note that the variety  $X$  in Totaro's proof is assumed to be smooth projective, while  $X = E^\times$  we use here is not projective despite being smooth. Nevertheless, the assumption that  $X$  is smooth and projective is to ensure that  $H^i(X^n)$  is pure of weight  $i$ . Since we have already checked  $H^i((E^\times)^n)$  is pure of weight  $i$  in Section V.2, we can apply the same argument. Totaro's description of the  $E_2$  page is given with respect to the **diagonal class**  $[\Delta] \in H^2(X^2)$  of  $X = E^\times$ . By definition, this is the image of  $1 \in H^0(X)$  via the composition

$$H^0(X) \simeq H_c^2(X)^\vee \rightarrow H_c^2(X^2)^\vee \simeq H^2(X^2), \quad (\text{V.3.6})$$

where the isomorphisms are given by Poincaré duality and the middle map is given by taking the dual of the pullback  $\delta^* : H_c^2(X^2) \rightarrow H_c^2(X)$  of the diagonal map  $\delta : X \hookrightarrow X^2$ . Following the maps above, we have

$$1 \mapsto \left( [\eta] \mapsto \int_X \eta \right) \mapsto \left( [\omega] \mapsto \int_X \delta^*(\omega) \right) = \left( [\omega] \mapsto \int_{X^2} \Delta \wedge \omega \right) \leftarrow [\Delta], \quad (\text{V.3.7})$$

so the diagonal class  $[\Delta]$  is characterized by the conditions

$$\int_X \delta^*(\omega) = \int_{X^2} \Delta \wedge \omega \quad (\text{V.3.8})$$

where  $\omega$  varies over all compactly supported 2-forms on  $X^2$  so that  $[\omega] \in H_c^2(X^2)$ . Our situation  $X = E^\times$  is special in the sense that  $[\Delta]$  can be explicitly computed in terms of two generators of  $H^1(X)$ . (Note that  $h^0(X) = 1$  and  $h^1(X) = 2$  while  $h^i(X) = 0$  for all  $i \geq 2$ .) We set some convenient notation to state this fact: denoting by  $p_1, p_2 : X^2 \rightarrow X$  the two projections, for any  $\alpha \in H^\bullet(X)$ , we write  $\alpha_i := p_i^*(\alpha) \in H^\bullet(X^2)$ .

**Lemma V.3.1.** *It is possible to choose a basis  $x, y \in H^1(X)$  such that*

$$[\Delta] = y_1 \wedge x_2 - x_1 \wedge y_2 \in H^2(X^2). \quad (\text{V.3.9})$$

*Proof.* We use the notation  $[\Delta_E]$  and  $[\Delta_X]$  to mean the diagonal classes of  $E$  and  $X = E^\times$ , respectively. First, it is well-known that there are closed 1-forms on  $E$  such that their cohomology classes  $x, y \in H^1(E, \mathbb{R}) \simeq \mathbb{R}^2$  form a basis for  $H^1(E, \mathbb{R}) \simeq \mathbb{R}^2$  and

$$\int_E x \wedge y = 1. \quad (\text{V.3.10})$$

We also use  $x, y$  to mean the basis  $x \otimes 1, y \otimes 1 \in H^1(E, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^2$ . These  $x, y$  are compactly supported in  $E$  because  $E$  is compact. We claim that

$$[\Delta_E] = x_1 \wedge y_1 + y_1 \wedge x_2 - x_1 \wedge y_2 + x_2 \wedge y_2 \quad (\text{V.3.11})$$

on  $E$ . To check this, one can just check such the right-hand side satisfies the equations

$$\int_E \delta^*(\omega) = \int_{E^2} \Delta_E \wedge \omega \quad (\text{V.3.12})$$

for all  $\omega \in H^2(E^2)$  in place of  $\Delta_E$ . Note that it is enough to show the identity for the following possibilities of  $\omega$ :

- $x_1 \wedge x_2$  and  $y_1 \wedge y_2$ ;
- $x_i \wedge y_j$  for  $1 \leq i, j \leq 2$ ,

as they form a basis for  $H^2(E^2)$ . Since  $p_i \circ \delta = \text{id}_X$  for  $i = 1, 2$ , we have  $\delta^*(x_1 \wedge x_2) = 0 =$

$\delta^*(y_1 \wedge y_2)$  and  $\delta^*(x_i \wedge y_j) = x \wedge y$ . On the other hand, we have

$$(x_1 \wedge y_1 + y_1 \wedge x_2 - x_1 \wedge y_2 + x_2 \wedge y_2) \wedge x_1 \wedge x_2 \quad (\text{V.3.13})$$

$$= 0 \quad (\text{V.3.14})$$

$$= (x_1 \wedge y_1 + y_1 \wedge x_2 - x_1 \wedge y_2 + x_2 \wedge y_2) \wedge y_1 \wedge y_2, \quad (\text{V.3.15})$$

and

$$(x_1 \wedge y_1 + y_1 \wedge x_2 - x_1 \wedge y_2 + x_2 \wedge y_2) \wedge x_i \wedge y_j = x_1 \wedge y_1 \wedge x_2 \wedge y_2. \quad (\text{V.3.16})$$

Since

$$\int_{E^2} x_1 \wedge y_1 \wedge x_2 \wedge y_2 = \int_{E^2} p_1^*(x \wedge y) \wedge p_2^*(x \wedge y) \quad (\text{V.3.17})$$

$$= \left( \int_E x \wedge y \right)^2 \quad (\text{V.3.18})$$

$$= 1, \quad (\text{V.3.19})$$

this establishes the claim.

To finish the proof, we first note that  $x_1 \wedge y_1$  and  $x_2 \wedge y_2$  pull back to zero on  $X^2 = (E^\times)^2$ . Thus, the claim follows from our computation on  $E$  because the restriction  $H^2(E^2) \rightarrow H^2(X^2)$  maps the diagonal class of  $E$  to that of  $X$  (by functoriality of Gysin maps). This finishes the proof.  $\square$

Let  $E_r^{p,q}(X, n)$  be the  $(p, q)$  component of the  $r$ -th page of the Leray spectral sequence of the inclusion  $F(X, n) \hookrightarrow X^n$  and

$$E_r(X, n) := \bigoplus_{i=0}^{\infty} \bigoplus_{p+q=i} E_r^{p,q}(X, n). \quad (\text{V.3.20})$$

Totaro's description of  $E_2(X, n)$  is as follows. We will only consider the case  $X = E^\times$ , but Totaro's result works for any oriented real manifold. However, the degeneration at the  $E_3$  page (i.e.,  $E_3(X, n) \simeq E_\infty(X, n)$ ) is particular to this case (or any variety whose  $i$ -th cohomology group is pure of weight  $i$  for every  $i \geq 0$ ).

**Proposition V.3.2** ([Tot96], Theorem 1). *Keeping the notation as above, we have*

$$E_2(X, n) = \frac{H^\bullet(X^n)[g_{ij} : 1 \leq i \neq j \leq n]}{(\text{relations})}, \quad (\text{V.3.21})$$

where the right-hand side is a presentation for the bigraded-commutative algebra with the following relations:

- (a)  $g_{ij} = g_{ji}$  for all  $1 \leq i \neq j \leq n$ ;
- (b)  $g_{ik}g_{jk} = -g_{ij}(g_{ik} + g_{jk})$  for all distinct  $1 \leq i, j, k \leq n$ ;
- (c)  $\alpha_i g_{ij} = \alpha_j g_{ij}$  for all  $1 \leq i \neq j \leq n$  and  $\alpha \in H^\bullet(X^n)$ ,

and the elements of  $H^p(X^n)$  get degree  $(p, 0)$  while each  $g_{ij}$  gets degree  $(0, 1)$ . The differential

$$d = d_2^{p,q} : E_2^{p,q}(X, n) \rightarrow E_2^{p+2,q-1}(X, n) \quad (\text{V.3.22})$$

is given by sending everything in  $H^\bullet(X^n)$  to 0 and

$$d(g_{ij}) = p_{ij}^*([\Delta]), \quad (\text{V.3.23})$$

where  $p_{ij} := (p_i, p_j) : X^n \rightarrow X^2$ . Moreover, the action of  $S_n$  on  $E_2(X, n)$  induced by permuting coordinates on  $F(X, n) \subset X^n$  is described as follows: the  $S_n$ -action on  $H^\bullet(X^n)$  is induced from the  $S_n$ -action on  $X^n$  by permuting coordinates and

$$\sigma g_{ij} := g_{\sigma(i), \sigma(j)} \quad (\text{V.3.24})$$

for  $\sigma \in S_n$ .

*Remark V.3.3.* By Lemma V.3.1, we necessarily have

$$d(g_{ij}) = y_i \wedge x_j - x_i \wedge y_j, \quad (\text{V.3.25})$$

where  $x_i := p_i^*(x)$  and  $y_i := p_i^*(y)$ , writing  $p_i : X^n \rightarrow X$  to mean the  $i$ -th projection. To see this, recall that Lemma V.3.1 says  $[\Delta] = y_1 \wedge x_2 - x_1 \wedge y_2$  so that

$$d(g_{ij}) = p_{ij}^*([\Delta]) \quad (\text{V.3.26})$$

$$= p_{ij}^*(y_1 \wedge x_2) - p_{ij}^*(x_1 \wedge y_2) \quad (\text{V.3.27})$$

$$= p_{ij}^*(y_1) \wedge p_{ij}^*(x_2) - p_{ij}^*(x_1) \wedge p_{ij}^*(y_2) \quad (\text{V.3.28})$$

$$= p_{ij}^*(p_1^*(y)) \wedge p_{ij}^*(p_2^*(x)) - p_{ij}^*(p_1^*(x)) \wedge p_{ij}^*(p_2^*(y)) \quad (\text{V.3.29})$$

$$= (p_1 \circ p_{ij})^*(y) \wedge (p_2 \circ p_{ij})^*(x) - (p_1 \circ p_{ij})^*(x) \wedge (p_2 \circ p_{ij})^*(y) \quad (\text{V.3.30})$$

$$= p_i^*(y) \wedge p_j^*(x) - p_i^*(x) \wedge p_j^*(y) \quad (\text{V.3.31})$$

$$= y_i \wedge x_j - x_i \wedge y_j. \quad (\text{V.3.32})$$

We have

$$E_3^{p,q}(X, n) = \text{Gr}_{p+2q}(H^{p+q}(F(X, n))), \quad (\text{V.3.33})$$

but following [Tot96], we also have

$$E_3^{p,q}(X, n)^{S_n} = \text{Gr}_{p+2q}(H^{p+q}(F(X, n)/S_n)) = \text{Gr}_{p+2q}(H^{p+q}(\text{Conf}^n(X))), \quad (\text{V.3.34})$$

and hence we can use the presentation of  $E_2(X, n)$  described above to approach Theorem V.1.3.

## V.4: Proof of Theorem V.1.3

In this section, we prove Theorem V.1.3. Again, we work with  $X = E^\times$ , a punctured elliptic curve over  $\mathbb{C}$ .

### V.4.1: Setup and goal

The first step is to note that our problem is entirely algebraic. That is, from the previous section, we have the following description of the graded-commutative  $\mathbb{C}$ -algebra:

$$E_2(X, n) = \mathbb{C} \left[ \begin{array}{c} x_1, \dots, x_n, \\ y_1, \dots, y_n, \\ g_{ij} \text{ for } 1 \leq i \neq j \leq n \end{array} \right] / (\text{relations}), \quad (\text{V.4.1})$$

where the relations are given by

- (a)  $g_{ij} = g_{ji}$  for all  $1 \leq i \neq j \leq n$ ;
- (b)  $g_{ik}g_{jk} = -g_{ij}(g_{ik} + g_{jk})$  for all distinct  $1 \leq i, j, k \leq n$ ;
- (c)  $g_{ij}x_i = g_{ij}x_j$  for all  $1 \leq i \neq j \leq n$ ;
- (d)  $g_{ij}y_i = g_{ij}y_j$  for all  $1 \leq i \neq j \leq n$ ;
- (e)  $x_iy_i = 0$  for all  $1 \leq i \leq n$ ,

and the degrees of  $x_i, y_i$  are  $(1, 0)$ , while the degree of each  $g_{ij}$  is  $(0, 1)$ .<sup>7</sup> By Remark V.3.3, the differential

$$d = d_2^{p,q} : E_2^{p,q}(X, n) \rightarrow E_2^{p+2, q-1}(X, n) \quad (\text{V.4.2})$$

---

<sup>7</sup>When it comes to graded-commutativity, the total degree of an element of degree  $(p, q)$  is  $p + q$ .

is given by

$$d(g_{ij}) = y_i x_j - x_i y_j. \quad (\text{V.4.3})$$

**Goal.** Proving Theorem V.1.3 is equivalent to proving that  $E_3^{p,q}(X, n)^{S_n} = 0$  unless  $p - q = 0$  or 1. Indeed, since  $p + q = i$  and  $p + 2q = w(i)$ , we have  $p = 2i - w(i)$  and  $q = w(i) - i$  so that  $p - q = 3i - 2w(i)$ . This implies that  $w(i) = (3i - (p - q))/2$ .

#### V.4.2: Special elements of $E_2(X, n)$

Since we are over  $\mathbb{C}$ , whose characteristic is 0, taking the  $S_n$ -invariants is an exact functor, so it commutes with taking cohomology. Thus, we may take the cohomology of  $E_2(X, n)^{S_n}$  to compute  $E_3(X, n)^{S_n}$ . A typical element of  $E_2(X, n)$  is a linear combination of the elements of the form

$$g_{i_1, j_1} \cdots g_{i_a, j_a} x_{k_1} \cdots x_{k_b} y_{l_1} \cdots y_{l_c}. \quad (\text{V.4.4})$$

We may assume that all of  $k_1, \dots, k_b, l_1, \dots, l_c$  are distinct because otherwise such an element is 0 by graded commutativity or some of the relations we have above. In this section, we provide various lemmas about such elements that helps us prove Theorem V.1.3.

**Lemma V.4.1.** *For any  $r \geq 2$ , we have*

$$g_{1,2} g_{2,3} \cdots g_{r-1,r} g_{r,1} = 0 \quad (\text{V.4.5})$$

in  $E_2(X, n)$ .

*Proof.* We proceed by induction. For  $r = 2$ , we have  $g_{1,2} g_{2,1} = g_{1,2}^2 = 0$  by the relation (1) and graded commutativity. For the induction hypothesis, suppose that  $g_{1,2} g_{2,3} \cdots g_{r-2,r-1} g_{r-1,1} = 0$  with  $r - 1 \geq 2$ . Then

$$g_{1,2} g_{2,3} \cdots g_{r-2,r-1} g_{r-1,r} g_{r,1} = g_{1,2} g_{2,3} \cdots g_{r-2,r-1} (-g_{r-1,1} (g_{r-1,r} + g_{r,1})) \quad (\text{V.4.6})$$

$$= -(g_{1,2} g_{2,3} \cdots g_{r-2,r-1} g_{r-1,1}) (g_{r-1,r} + g_{r,1}) \quad (\text{V.4.7})$$

$$= 0 \quad (\text{V.4.8})$$

by (2) and the induction hypothesis. This finishes the proof.  $\square$

The following notation is convenient for our proof:

- $x_{ij} := g_{ij} x_i = g_{ij} x_j$ ;
- $y_{ij} := g_{ij} y_i = g_{ij} y_j$ ;

- $g_I = g_{i_1, \dots, i_r} := g_{i_1, i_2} g_{i_2, i_3} \cdots g_{i_{r-2}, i_{r-1}} g_{i_{r-1}, i_r}$ ;
- $x_I = x_{i_1, \dots, i_r} := g_{i_1, \dots, i_r} x_{i_1} = \cdots = g_{i_1, \dots, i_r} x_{i_r}$ ;
- $y_I = y_{i_1, \dots, i_r} := g_{i_1, \dots, i_r} y_{i_1} = \cdots = g_{i_1, \dots, i_r} y_{i_r}$ ,

where  $I = (i_1, \dots, i_r)$  and  $i_1, \dots, i_r$  are distinct.

**Lemma V.4.2.** *Any product of  $g_{i,j}$  can be written as a linear combination of elements in  $E_2(X, n)$  of the following form:*

$$g_{I_1} g_{I_2} \cdots g_{I_r} \tag{V.4.9}$$

where  $I_1, \dots, I_r$  are disjoint ordered tuples.

*Proof.* Given a product of  $g_{i,j}$ , consider the undirected graph with vertex set  $\{1, \dots, n\}$  with  $i, j$  connected by an edge if and only if  $g_{i,j}$  appears in the product. We may disregard the vertices that have no edges connected to them. By Lemma V.4.1, we may assume that this graph is acyclic, so it is a disjoint union of trees. Consider the following special case first:  $g_{1,2} g_{2,3} g_{2,4}$ . We have

$$g_{1,2} g_{2,3} g_{2,4} = g_{1,2} (-g_{3,4} (g_{2,3} + g_{2,4})) \tag{V.4.10}$$

$$= -g_{1,2} g_{3,4} g_{2,3} - g_{1,2} g_{3,4} g_{2,4} \tag{V.4.11}$$

$$= g_{1,2} g_{2,3} g_{3,4} + g_{1,2} g_{2,4} g_{4,3} \tag{V.4.12}$$

$$= g_{1,2,3,4} + g_{1,2,4,3}. \tag{V.4.13}$$

For the general case, applying this computation repeatedly to each connected component finishes the proof. (There are several ways to show that the process terminates on a given tree. For instance, one can assign a root and proceed by induction on the depth of the tree, namely, the length of the longest path from the root.)  $\square$

By Lemma V.4.2, we may write any typical element of  $E_2(X, n)$  as a linear combinations of the elements of the form

$$g_{I_1} \cdots g_{I_a} x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c}, \tag{V.4.14}$$

where  $I_1, \dots, I_a, J_1, \dots, J_b, K_1, \dots, K_c$  are disjoint ordered sets of distinct integers. Since

$E_2(X, n)^{S_n} = e_{S_n}(E_2(X, n))$ , where

$$e_{S_n} := \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma \quad (\text{V.4.15})$$

is the averaging operator, we can list all the generators of  $E_2(X, n)^{S_n}$  by applying  $e_{S_n}$  to the elements of the above form. We note that

$$d(x_{ij}) = d(g_{ij})x_i - g_{ij}d(x_i) = d(g_{ij})x_i = (y_i x_j - x_i y_j)x_i = 0 \quad (\text{V.4.16})$$

because  $x_i y_i = 0$  and  $x_i^2 = 0$ . Similarly, we have  $d(y_{ij}) = 0$ .

**Lemma V.4.3.** *Let*

$$\alpha = g_{I_1} \cdots g_{I_a} x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c} \quad (\text{V.4.17})$$

be an element of  $E_2(X, n)$ , where  $I_1, \dots, I_a, J_1, \dots, J_b, K_1, \dots, K_c$  are disjoint ordered sets of distinct integers.

(a) *If  $\alpha$  includes  $x_i$  and  $x_j$  for some  $1 \leq i \neq j \leq n$ , then  $e_{S_n}(\alpha) = 0$ .*

(b) *If  $\alpha$  includes  $g_{ij}$  and  $g_{kl}$  such that  $\{i, j\} \cap \{k, l\} = \emptyset$ , then  $e_{S_n}(\alpha) = 0$ .*

*Proof.* If  $\alpha$  includes  $x_i$  and  $x_j$  for some  $1 \leq i \neq j \leq n$ , then we consider the transposition  $\sigma = (i \ j) \in S_n$  and see that  $e_{S_n}(\alpha) = e_{S_n}(\sigma(\alpha)) = e_{S_n}(-\alpha) = -e_{S_n}(\alpha)$  so that  $e_{S_n}(\alpha) = 0$  since we are working over  $\mathbb{C}$ , a field of characteristic 0. This proves the first assertion. The second assertion can be similarly proven by taking  $\sigma = (i \ k)(j \ l)$  instead.  $\square$

**Lemma V.4.4.** *For  $r \geq 3$ , we have*

$$e_{S_n}(g_{i_1, \dots, i_r}) = 0. \quad (\text{V.4.18})$$

*Proof.* By a result of Cohen (e.g., [Coh88], Section 6), we have a graded isomorphism of graded-commutative  $\mathbb{C}[S_n]$ -algebras

$$H^\bullet(F(\mathbb{A}^1, n)) \simeq \mathbb{C} \left[ g_{ij} \text{ for } 1 \leq i \neq j \leq n \right] / (\text{relations}), \quad (\text{V.4.19})$$

where the relations are given by

(a)  $g_{ij} = g_{ji}$  for all  $1 \leq i \neq j \leq n$ ;

(b)  $g_{ik}g_{jk} = -g_{ij}(g_{ik} + g_{jk})$  for all distinct  $1 \leq i, j, k \leq n$ ,



and the degree of each  $g_{ij}$  is 1. Cohen's result also says that the  $S_n$ -action induced from the one on  $F(\mathbb{A}^1, n) \subset \mathbb{A}^n$ , by permuting coordinates, is given by  $\sigma g_{i,j} = g_{\sigma(i),\sigma(j)}$ . We thus have a map

$$H^\bullet(F(\mathbb{A}^1, n)) \rightarrow E_2(X, n) \quad (\text{V.4.20})$$

of  $\mathbb{C}[S_n]$ -algebras given by  $g_{ij} \mapsto g_{ij}$ . Taking the  $S_n$ -invariant is an exact functor, so we get

$$H^\bullet(\text{Conf}^n(\mathbb{A}^1)) \simeq H^\bullet(F(\mathbb{A}^1, n))^{S_n} \rightarrow E_2(X, n)^{S_n}, \quad (\text{V.4.21})$$

where the isomorphism is given because  $F(\mathbb{A}^1, n) \rightarrow \text{Conf}^n(\mathbb{A}^1)$  is a finite covering space. Consider the element  $e_{S_n}(g_{i_1, \dots, i_r})$  in  $H^{r-1}(\text{Conf}^n(\mathbb{A}^1))$ . For  $r \geq 3$ , Arnol'd [Arn69] proved that  $H^{r-1}(\text{Conf}^n(\mathbb{A}^1)) = 0$  for all  $n \geq 0$ . Thus, we must have  $e_{S_n}(g_{i_1, \dots, i_r}) = 0$  in  $H^{r-1}(\text{Conf}^n(\mathbb{A}^1))$ , as well as in  $E_2(X, n)^{S_n}$ .  $\square$

**Lemma V.4.5.** *Every element of  $E_2(X, n)^{S_n}$  is a linear combination of elements of the form*

$$e_{S_n}(g_{i_1, i_2}^r x_j^{s_1} y_k^{s_2} x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c}) \in E_2(X, n), \quad (\text{V.4.22})$$

where  $r, s_1, s_2 \in \{0, 1\}$ , the tuples  $J_1, \dots, J_b, K_1, \dots, K_c$  all have size 2, and all the lower indices that appear are disjoint. (For example, if  $s_1 = 0$ , then the index  $j$  does not appear.)

*Proof.* Since  $E_2(X, n)^{S_n} = e_{S_n}(E_2(X, n)) \subset E_2(X, n)$ , by Lemma V.4.3, any element of it can be written as a linear combination of elements of the form  $e_{S_n}(\alpha)$  such that

$$\alpha = g_{I_1} \cdots g_{I_a} x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c}, \quad (\text{V.4.23})$$

where  $I_1, \dots, I_a, J_1, \dots, J_b, K_1, \dots, K_c$  are disjoint ordered sets of distinct integers. If any  $I_t$  has size at least 3, then  $e_{S_n}(\alpha) = 0$  because of mutual disjointness of the index sets and Lemma V.4.4. Thus, if there are any  $g_{I_t}$  in the expression, we may assume that  $|I_t| = 2$ .

If  $\sigma$  is any permutation on some restricted letters, say  $1, 2, \dots, r$ , then

$$\sigma x_{1,2,\dots,r} = \sigma(g_{1,2,\dots,r} x_1) = g_{\sigma(1),\sigma(2),\dots,\sigma(r)} x_{\sigma(1)} = x_{\sigma(1),\sigma(2),\dots,\sigma(r)}. \quad (\text{V.4.24})$$

Thus, we may apply a similar argument as before to be able to assume that

$$|J_1|, \dots, |J_b|, |K_1|, \dots, |K_c| \leq 2 \quad (\text{V.4.25})$$

in order for  $e_{S_n}(\alpha)$  to be nonzero.

Finally, by Lemma V.4.3, we have that  $e_{S_n}(\alpha)$  is zero unless  $a \leq 1$ , there is at most one  $J_t$  with  $|J_t| = 1$ , and there is at most one  $K_t$  with  $|K_t| = 1$ . This finishes the proof.  $\square$

*Remark V.4.6.* Assume  $|I|, |J|, |K| = 2$ . In our notation for  $g_I, x_J$ , and  $y_K$ , even though the definitions require  $I, J$ , and  $K$  to be ordered sets of integers, the orders do not matter due to some of the relations we described at the beginning of Section V.4.1. Hence, from now on, we may consider  $I, J$ , and  $K$  as unordered sets of size 2 without ambiguity.

We also need the following technical results in the proof of Theorem V.1.3.

**Lemma V.4.7.** *Let  $n, b, c \in \mathbb{Z}_{\geq 0}$  such that  $n \geq 2 + b + c$ . The elements of the form*

$$x_j y_k x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c}, \quad (\text{V.4.26})$$

where  $\{j, k\}, J_1, \dots, J_b, K_1, \dots, K_c$  are disjoint subsets of  $\{1, 2, \dots, n\}$  and  $|J_t| = |K_u| = 2$  for  $1 \leq t \leq b$  and  $1 \leq u \leq c$ , with the increasing lexicographic order among the two-element set indices,<sup>8</sup> are linearly independent in  $E_2(X, n)$ .

**Corollary V.4.8.** *With the notation in Lemma V.4.7, we have*

$$e_{S_n}(x_j y_k x_{J_1} \cdots x_{J_b} y_{K_1} \cdots y_{K_c}) \neq 0 \quad (\text{V.4.27})$$

in  $E_2(X, n)^{S_n}$ .

*Proof of Corollary V.4.8 given Lemma V.4.7.* Since we work over a field of characteristic 0, it is enough to show that

$$\sum_{\sigma \in S_n} x_{\sigma(j)} y_{\sigma(k)} x_{\sigma(J_1)} \cdots x_{\sigma(J_b)} y_{\sigma(K_1)} \cdots y_{\sigma(K_c)} \neq 0. \quad (\text{V.4.28})$$

This follows from Lemma V.4.7 because any elements of the form  $x_J$  or  $y_K$  with two-element indices  $J, K$  commute.  $\square$

We now prove Lemma V.4.7:

*Proof of Lemma V.4.7.* Consider the following graded-commutative  $\mathbb{C}$ -algebra

$$W(n) := E_2(X, n) / (g_{ij} g_{jk} \text{ for } i, j, k \text{ distinct}).$$

Using the discussion in Section V.4.1, we see that the algebra  $W(n)$  has the following presentation:

$$W(n) = \mathbb{C} \left[ \begin{array}{c} x_1, \dots, x_n, \\ y_1, \dots, y_n, \\ g_{ij} \text{ for unordered } i, j \text{ with } 1 \leq i \neq j \leq n \end{array} \right] / (\text{relations}) \quad (\text{V.4.29})$$

---

<sup>8</sup>For example, we have  $\{1, 2\} < \{3, 6\} < \{4, 5\}$ .

with  $\deg(x_i) = \deg(y_i) = \deg(g_{ij}) = 1$ , and  $g_{ij}$  and  $g_{ji}$  are treated as the same generator<sup>9</sup> and the relations are given by

- (a)  $g_{ij}g_{jk} = 0$  for all distinct  $1 \leq i, j, k \leq n$ ;
- (b)  $g_{ij}x_i = g_{ij}x_j$  for all  $1 \leq i \neq j \leq n$ ;
- (c)  $g_{ij}y_i = g_{ij}y_j$  for all  $1 \leq i \neq j \leq n$ ;
- (d)  $x_iy_i = 0$  for all  $1 \leq i \leq n$ .

To establish a linear independence in  $E_2(X, n)$ , it is sufficient to check the linear independence in  $W(n)$ . The strategy is to compare  $W(n)$  with the following algebra for which we know a basis.

Consider the graded-commutative  $\mathbb{C}$ -algebra

$$V(n) := \mathbb{C} \left[ \begin{array}{c} x_1, \dots, x_n, \\ y_1, \dots, y_n, \\ x_{ij}, y_{ij} \text{ for unordered } 1 \leq i \neq j \leq n \end{array} \right] / (\text{relations}) \quad (\text{V.4.30})$$

with  $\deg(x_i) = \deg(y_i) = 1$  and  $\deg(x_{ij}) = \deg(y_{ij}) = 2$ , where the relations are generated by any monomial with repeated indices (e.g.,  $x_1y_{12}$  or  $x_1y_2x_{34}x_{56}y_{57}$ ). We have a map  $\phi : V(n) \rightarrow W(n)$  of graded-commutative  $\mathbb{C}$ -algebras given by  $x_i \mapsto x_i$ ,  $y_i \mapsto y_i$ ,  $x_{ij} \mapsto g_{ij}x_i$ , and  $y_{ij} \mapsto g_{ij}y_i$ . Because the relations defining  $V(n)$  are generated by monomials, it can be easily checked that a  $\mathbb{C}$ -vector space basis for  $V(n)$  can be given by all monomials with disjoint indices (up to rearrangement), so to finish our proof, it is enough to show that the map  $\phi : V(n) \rightarrow W(n)$  is injective. We will achieve this by constructing a left inverse  $\psi : W(n) \rightarrow V(n)$  as  $\mathbb{C}$ -vector spaces, although this map will not be a map of  $\mathbb{C}$ -algebras.

Consider

$$\widetilde{W}(n) := \mathbb{C} \left[ \begin{array}{c} x_1, \dots, x_n, \\ y_1, \dots, y_n, \\ g_{ij} \text{ for unordered } 1 \leq i \neq j \leq n \end{array} \right], \quad (\text{V.4.31})$$

a free graded-commutative  $\mathbb{C}$ -algebra with  $\deg(x_i) = \deg(y_i) = \deg(g_{ij}) = 1$ . As a  $\mathbb{C}$ -vector space, a basis of  $\widetilde{W}(n)$  can be given by all monomials

$$g_{I_1} \cdots g_{I_a} x_{j_1} \cdots x_{j_b} y_{k_1} \cdots y_{j_c}, \quad (\text{V.4.32})$$

---

<sup>9</sup>We are writing  $g_{ij} := g_{\{i,j\}}$  to mean the formal variable corresponding to each two-element subset  $\{i, j\} \subset \{1, 2, \dots, n\}$ .

such that

- $I_1, \dots, I_a$  are in the increasing lexicographic order,
- $j_1, \dots, j_b$  are increasing, and
- $k_1, \dots, k_c$  are increasing.

We define  $\tilde{\psi} : \widetilde{W}(n) \rightarrow V(n)$  by mapping each of such monomials to an element in  $V(n)$  according to the following rules, which we mark as (A), (B), and (C):

**(A)** The map  $\tilde{\psi}$  sends the monomial to 0 if it contains any of the factors of the following forms (possibly after a rearrangement):  $g_{ij}g_{jk}, x_i y_i, g_{ij}x_i x_j, g_{ij}y_i y_j, g_{ij}x_i y_j$ . We say such monomials are of **type A**.

Any nonzero monomial that does not fall into any of the above cases satisfies the following: if  $g_{ij}$  appears as a factor, then at most one of  $x_i, x_j, y_i, y_j$  can appear; if  $x_i$  or  $y_i$  appears, then at most one  $g_I$  such that  $i \in I$  can appear. As a result, such a monomial can be uniquely written in the following form (possibly after a rearrangement and a change of sign):

$$(g_{I_1} \cdots g_{I_a})(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c})[(g_{L_1}x_{l_1}) \cdots (g_{L_d}x_{l_d})][(g_{M_1}y_{m_1}) \cdots (g_{M_e}y_{m_e})], \quad (\text{V.4.33})$$

where  $I_1, \dots, I_a, j_1, \dots, j_b, k_1, \dots, k_c, L_1, \dots, L_d, M_1, \dots, M_e$  are disjoint while

- $I_1, \dots, I_a$  are in the increasing lexicographic order,
- $j_1, \dots, j_b$  are increasing,
- $k_1, \dots, k_c$  are increasing,
- $L_1, \dots, L_d$  are in the increasing lexicographic order, and
- $M_1, \dots, M_e$  are in the increasing lexicographic order,

with  $l_s \in L_s$  for  $1 \leq s \leq d$  and  $m_t \in M_t$  for  $1 \leq t \leq e$ .

**(B)** If  $a > 0$ , then  $\tilde{\psi}$  sends the above monomial to 0. Such a monomial is said to be of **type B**.

**(C)** If  $a = 0$ , then  $\tilde{\psi}$  sends the above monomial to

$$(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c})(x_{L_1} \cdots x_{L_d})(y_{M_1} \cdots y_{M_e}) \in V(n).$$

Such a monomial is said to be of **type C**. This finishes the construction of  $\tilde{\psi} : \widetilde{W}(n) \rightarrow V(n)$ .

We claim that  $\tilde{\psi}$  factors through a  $\mathbb{C}$ -linear map  $\psi : W(n) \rightarrow V(n)$ . That is, we want to show that the elements of the following forms go to 0 under  $\tilde{\psi}$ :

- (a)  $g_{ij}g_{jk}M$  (a type A monomial);
- (b)  $g_{ij}(x_i - x_j)M$ ;
- (c)  $g_{ij}(y_i - y_j)M$ ;
- (d)  $x_i y_i M$  (a type A monomial),

where  $M = M(\mathbf{x}, \mathbf{y}, \mathbf{g})$  is a monomial. It is immediate from the rule (A) of the definition of  $\tilde{\psi}$  that

$$\tilde{\psi}(g_{ij}g_{jk}M) = 0 = \tilde{\psi}(x_i y_i M). \quad (\text{V.4.34})$$

To show  $\tilde{\psi}(g_{ij}(x_i - x_j)M) = 0$ , we may show that

$$\tilde{\psi}(g_{ij}x_i M) = \tilde{\psi}(g_{ij}x_j M) \quad (\text{V.4.35})$$

If  $M$  contains any one of  $x_i, x_j, y_i, y_j$ , or  $g_I$  with  $I \cap \{i, j\} \neq \emptyset$ , then both sides are 0 according to the graded-commutativity and the rule (A), which gives the equality. Thus, we may assume  $M$  involves only indices other than  $i$  and  $j$ . Since  $g_{ij}x_i M$  and  $g_{ij}x_j M$  are both monomials, if  $M$  is a monomial of type A, then both sides of (V.4.35) are 0 according to the rule (A). This will finish our task, so suppose that this is not the case. That is, the monomial  $M$  can be written in the form of (V.4.33) (possibly after a rearrangement and a change of sign):

$$M = (g_{I_1} \cdots g_{I_a})(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c})[(g_{L_1}x_{l_1}) \cdots (g_{L_d}x_{l_d})][(g_{M_1}y_{m_1}) \cdots (g_{M_e}y_{m_e})], \quad (\text{V.4.36})$$

where the indices are disjoint from  $\{i, j\}$ . It follows that both  $g_{ij}x_i M$  and  $g_{ij}x_j M$  can be written in the form of (V.4.33) as

$$g_{ij}x_i M = (g_{I_1} \cdots g_{I_a})(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c}) \quad (\text{V.4.37}) \\ [(g_{L_1}x_{l_1}) \cdots (g_{L_{d^*}}x_{l_{d^*}})(g_{ij}x_i)(g_{L_{d^*+1}}x_{l_{d^*+1}}) \cdots (g_{L_d}x_{l_d})][(g_{M_1}y_{m_1}) \cdots (g_{M_e}y_{m_e})],$$

and

$$g_{ij}x_j M = (g_{I_1} \cdots g_{I_a})(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c}) \quad (\text{V.4.38}) \\ [(g_{L_1}x_{l_1}) \cdots (g_{L_{d^*}}x_{l_{d^*}})(g_{ij}x_j)(g_{L_{d^*+1}}x_{l_{d^*+1}}) \cdots (g_{L_d}x_{l_d})][(g_{M_1}y_{m_1}) \cdots (g_{M_e}y_{m_e})],$$

where  $d^*$  satisfies that  $L_1, \dots, L_{d^*}, \{i, j\}, L_{d^*+1}, \dots, L_d$  are in the increasing lexicographic order. Note that  $g_{ij}x_i$  and  $g_{ij}x_j$  have degree 2, so they commute with every element in  $\tilde{W}(n)$ .

If  $a > 0$ , then both sides of (V.4.35) are 0 according to the rule (B). If  $a = 0$ , then  $\tilde{\psi}$  sends both  $g_{ij}x_iM$  and  $g_{ij}x_jM$  to

$$(x_{j_1} \cdots x_{j_b})(y_{k_1} \cdots y_{k_c})(x_{L_1} \cdots x_{L_{d^*}} x_{ij} x_{L_{d^*+1}} x_{L_d})(y_{M_1} \cdots y_{M_e}) \in V(n)$$

according to the rule (C), so both sides of (V.4.35) are equal. This shows  $\tilde{\psi}(g_{ij}(x_i - x_j)M) = 0$ , and a similar argument shows  $\tilde{\psi}(g_{ij}(y_i - y_j)M) = 0$ . Therefore, we get a map  $\psi : W(n) \rightarrow V(n)$ , and it is left to show that  $\psi \circ \phi$  is the identity map on  $V(n)$ . It suffices to show that  $\psi(\phi(M)) = M$  for any monomial  $M$  in  $V(n)$ , but this is immediate.  $\square$

### V.4.3: Proof of Theorem V.1.3

Due to Lemma V.4.5, we now know that elements of  $E_2(X, n)^{S_n}$  can be written as linear combinations of elements of the form  $e_{S_n}(\alpha)$ , where

$$\alpha = g_{i_1, i_2}^r x_j^{s_1} y_k^{s_2} x_{j_1, 1, j_1, 2} \cdots x_{j_{t_1, 1}, j_{t_1, 2}} y_{k_{1, 1}, k_{1, 2}} \cdots y_{k_{t_2, 1}, k_{t_2, 2}} \in E_2(X, n), \quad (\text{V.4.39})$$

with  $r, s_1, s_2 \in \{0, 1\}$ , and all the numbers appearing as indices are distinct. We note that  $\alpha \in E_2^{p, q}(X, n)$ , where

$$(p, q) = (0, r) + (s_1 + s_2, 0) + (2(t_1 + t_2), 2(t_1 + t_2)) \quad (\text{V.4.40})$$

$$= (s_1 + s_2 + 2(t_1 + t_2), r + 2(t_1 + t_2)), \quad (\text{V.4.41})$$

so  $p - q = s_1 + s_2 - r$ .

*Proof of Theorem V.1.3.* We continue the discussion above. Our goal is to show that  $E_3^{p, q}(X, n)^{S_n} = 0$  unless  $p - q = s_1 + s_2 - r \in \{0, 1\}$ , where we also recall

- $p = s_1 + s_2 + 2(t_1 + t_2)$  and
- $q = r + 2(t_1 + t_2)$ .

Note that

$$d(\alpha) = d(g_{i_1, i_2}^r x_j^{s_1} y_k^{s_2} x_{j_1, 1, j_1, 2} \cdots x_{j_{t_1, 1}, j_{t_1, 2}} y_{k_{1, 1}, k_{1, 2}} \cdots y_{k_{t_2, 1}, k_{t_2, 2}}) \quad (\text{V.4.42})$$

$$= d(g_{i_1, i_2}^r x_j^{s_1} y_k^{s_2} x_{j_1, 1, j_1, 2} \cdots x_{j_{t_1, 1}, j_{t_1, 2}} y_{k_{1, 1}, k_{1, 2}} \cdots y_{k_{t_2, 1}, k_{t_2, 2}}) \quad (\text{V.4.43})$$

$$= \begin{cases} 0 & \text{if } r = 0, \\ (y_{i_1} x_{i_2} - x_{i_1} y_{i_2}) x_j^{s_1} y_k^{s_2} x_{j_1, 1, j_1, 2} \cdots x_{j_{t_1, 1}, j_{t_1, 2}} y_{k_{1, 1}, k_{1, 2}} \cdots y_{k_{t_2, 1}, k_{t_2, 2}} & \text{if } r = 1. \end{cases} \quad (\text{V.4.44})$$

This implies that

$$d(e_{S_n}(\alpha)) = e_{S_n}(d(\alpha)) \tag{V.4.45}$$

$$= \begin{cases} e_{S_n}((y_{i_1}x_{i_2} - x_{i_1}y_{i_2})x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}} \\ \quad y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}}) & \text{if } (r, s_1, s_2) = (1, 0, 0), \\ 0 & \text{otherwise,} \end{cases} \tag{V.4.46}$$

$$= \begin{cases} -2e_{S_n}(x_{i_1}y_{i_2}x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}}y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}}) & \text{if } (r, s_1, s_2) = (1, 0, 0), \\ 0 & \text{otherwise.} \end{cases} \tag{V.4.47}$$

By Corollary V.4.8, this shows that  $d(e_{S_n}(\alpha)) \neq 0$  if and only if  $(r, s_1, s_2) = (1, 0, 0)$ . Thus, we see that  $\ker(d)$ , where  $d$  is now the differential for  $E_2(X, n)^{S_n}$ , is generated by  $e_{S_n}(\alpha)$  for  $\alpha$  with  $(r, s_1, s_2) \neq (1, 0, 0)$ .

The table

$(r, s_1, s_2)$	$p - q = s_1 + s_2 - r$
(0,0,0)	0
(0,0,1)	1
(0,1,0)	1
(0,1,1)	2
(1,0,0)	-1
(1,0,1)	0
(1,1,0)	0
(1,1,1)	1

tells us that the only two choices for  $(r, s_1, s_2)$  that give  $p - q = s_1 + s_2 - r \notin \{0, 1\}$  are  $(1, 0, 0)$  and  $(0, 1, 1)$ .

First, let us consider the case  $(r, s_1, s_2) = (1, 0, 0)$ , which corresponds to  $p - q = -1$ . Then  $e_{S_n}(\alpha) \notin \ker(d)$ , and since  $\ker(d)$  is bigraded, this implies that  $\ker(d^{p,q}) = 0$  for any  $p, q$  such that  $p - q = -1$ . This implies that  $E_3^{p,q}(X, n)^{S_n} = 0$  for any such  $p, q$ .

Next, we consider the case where  $\alpha$  has  $(r, s_1, s_2) = (0, 1, 1)$ . Then

$$e_{S_n}(\alpha) = -2^{-1}(-2)e_{S_n}(x_j y_k x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}} y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}}) \tag{V.4.48}$$

$$= -2^{-1}e_{S_n}((y_j x_k - x_j y_k)x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}} y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}}) \tag{V.4.49}$$

$$= -2^{-1}e_{S_n}(d(g_{j,k}x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}} y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}})) \tag{V.4.50}$$

$$= d(e_{S_n}(-2^{-1}(g_{j,k}x_{j_{1,1},j_{1,2}} \cdots x_{j_{t_1,1},j_{t_1,2}} y_{k_{1,1},k_{1,2}} \cdots y_{k_{t_2,1},k_{t_2,2}}))). \tag{V.4.51}$$

Therefore, we have  $[e_{S_n}(\alpha)] = 0$  in  $E_3(X, n)^{S_n}$ . This shows that  $E_3^{p,q}(X, n)^{S_n} = 0$  unless  $p - q \in \{0, 1\}$ , as desired.  $\square$

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# CHAPTER VI

## Cohomology of Configuration Spaces on Punctured Varieties

The content of this chapter is available at [\[Hua20a\]](#).

### Abstract

We describe how the rational cohomology groups of unordered configuration spaces are affected by puncturing a point from the base space, by giving an identity involving Betti numbers that holds whenever the base space is a smooth noncompact complex variety satisfying a flexible mixed-Hodge-theoretic assumption. This extends certain results of Napolitano and Kallel. We also obtain a similar identity involving mixed Hodge numbers, which refines the identity about Betti numbers. Furthermore, we refine the above results to an equivariant theorem about the cohomology of ordered configuration spaces. Our approach involves explicit algebraic computations based on an Orlik–Solomon spectral sequence.

### VI.1: Introduction

#### VI.1.1: History and motivation

Let  $X$  be a topological space. Define the  $n$ -th *ordered configuration space* of  $X$  as the complement

$$F(X, n) := X^n - \bigcup_{i < j} \{(x_1, \dots, x_n) \in X^n : x_i = x_j\} \quad (\text{VI.1.1})$$

of the big diagonal in the product space, and define the  $n$ -th *unordered configuration space* as a topological quotient

$$\text{Conf}^n(X) := F(X, n)/S_n \quad (\text{VI.1.2})$$

by the symmetric group  $S_n$  acting by permuting coordinates. If  $X$  is a quasiprojective variety over a field, then  $F(X, n)$  is defined by the same formula, understood as an open subvariety

of  $X^n$ , and  $\text{Conf}^n(X)$  is the scheme-theoretic quotient of  $F(X, n)$  by the natural action of  $S_n$ . Both  $F(X, n)$  and  $\text{Conf}^n(X)$  are quasiprojective varieties as well.

Configuration spaces form a playground where topology and number theory often interplay. As a notable example, Church, Ellenberg and Farb [CEF14] pointed out and explained the connection between the Betti numbers of configuration spaces of the complex plane  $\mathbb{C}$  and the counting of square-free polynomials over a finite field. Let  $\mathbb{A}_{\mathbb{F}_q}^1$  denote the affine line over the finite field  $\mathbb{F}_q$  with  $q$  elements. Church, Ellenberg and Farb realized that the set of  $\mathbb{F}_q$ -points of the  $\mathbb{F}_q$ -variety  $\text{Conf}^n(\mathbb{A}_{\mathbb{F}_q}^1)$  can be identified with the set of degree- $n$  monic square-free  $\mathbb{F}_q$ -polynomials, and used this connection to show the identity

$$|\text{Conf}^n(\mathbb{A}_{\mathbb{F}_q}^1)(\mathbb{F}_q)| = \sum_{i=0}^{\infty} (-1)^i h^i(\text{Conf}^n(\mathbb{C})) q^{n-i}, \quad (\text{VI.1.3})$$

where the left-hand side is the number of monic square-free  $\mathbb{F}_q$ -polynomials of degree  $n$ , and the right-hand side involves  $h^i(\text{Conf}^n(\mathbb{C}))$ , the  $i$ -th (rational) Betti number of the configuration space  $\text{Conf}^n(\mathbb{C})$ .

The identity (VI.1.3) compares the following two aspects about the affine line  $\mathbb{A}^1$ : the cohomology of its configuration spaces over  $\mathbb{C}$ , and the point counts of its configuration spaces over  $\mathbb{F}_q$ . While such a connection does not exist for a variety in general, other phenomena do exist that occur in parallel in these two aspects. The focus of this paper is a “splitting” formula that compares the cohomology of configuration spaces of a topological space  $X$  with the configuration spaces of  $X$  minus one point. The study of such a splitting dates back to Fred Cohen [Coh93] and Gorjunov [Gor81].

Suppose  $X$  is a connected noncompact topological surface without boundary and  $P$  is a point of  $X$ . Napolitano [Nap03] showed that

$$H^i(\text{Conf}^n(X - P); \mathbb{Z}) \cong \bigoplus_{t=0}^{\infty} H^{i-t}(\text{Conf}^{n-t}(X); \mathbb{Z}). \quad (\text{VI.1.4})$$

In particular, this formula determines the Betti numbers of  $\text{Conf}^n(X - P)$  from the Betti numbers of  $\text{Conf}^m(X - P)$ ,  $m \leq n$ .

He noted that (VI.1.4) does not hold if  $X$  is compact, even in field coefficients. In fact, if  $X$  is a connected closed orientable surface of genus  $g$  and  $P$  is a point of  $X$ , then a different formula holds:

$$H^i(\text{Conf}^n(X); \mathbb{Z}/2\mathbb{Z}) = H^i(\text{Conf}^n(X - P); \mathbb{Z}/2\mathbb{Z}) \oplus H^{i-2}(\text{Conf}^{n-1}(X - P), \mathbb{Z}/2\mathbb{Z}). \quad (\text{VI.1.5})$$

Kallel [Kal08] extended the splitting (VI.1.4) in field coefficients to higher dimensions.

Let  $X$  be a connected noncompact manifold of even dimension  $2d$ , and let  $P$  be a point of  $X$ . Kallel showed that the following formula holds for any coefficient field  $\mathbb{F}$  if  $X$  can be obtained from removing  $r \geq 1$  points from a connected closed orientable manifold:

$$H^i(\mathrm{Conf}^n(X - P); \mathbb{F}) \cong \bigoplus_{t=0}^{\infty} H^{i-(2d-1)t}(\mathrm{Conf}^{n-t}(X); \mathbb{F}). \quad (\text{VI.1.6})$$

If  $X$  is a several-punctured orientable surface, then  $d = 1$ , so (VI.1.6) recovers the field-coefficient version of (VI.1.4). If  $X$  is a puncturing of a connected smooth compact variety over  $\mathbb{C}$  of (complex) dimension  $d$ , then the formula (VI.1.6) holds, and the special case  $\mathbb{F} = \mathbb{Q}$  implies the following identity of power series in  $u$  and  $t$ :

$$\sum_{i,n \geq 0} h^i(\mathrm{Conf}^n(X - P))(-u)^i t^n = \frac{1}{1 + u^{2d-1}t} \sum_{i,n \geq 0} h^i(\mathrm{Conf}^n(X))(-u)^i t^n. \quad (\text{VI.1.7})$$

Notably, an analogous formula also holds in terms of point counting over finite fields. Let  $X$  be any variety over  $\mathbb{F}_q$  and let  $P$  be an  $\mathbb{F}_q$ -point of  $X$ . As a consequence of a result due to Vakil and Wood [VW15, Proposition 5.9], we have

$$\sum_{n=0}^{\infty} |\mathrm{Conf}^n(X - P)(\mathbb{F}_q)| t^n = \frac{1}{1 + t} \sum_{n=0}^{\infty} |\mathrm{Conf}^n(X)(\mathbb{F}_q)| t^n. \quad (\text{VI.1.8})$$

The main contribution of this paper is to provide a version of (VI.1.7) and (VI.1.8) in terms of mixed Hodge numbers. Our formula refines both (VI.1.7) and a consequence of (VI.1.8) in a uniform way, which explains the analogy between (VI.1.7) and (VI.1.8). We also extend Kallel's formula (VI.1.7) to more general families of smooth noncompact complex varieties.

### VI.1.2: Main result

To state the main theorem and explain how it connects (VI.1.7) and (VI.1.8), we recall some necessary concepts in the mixed Hodge theory (for detailed references, see Deligne's [Del75] or [Del71]). For any complex variety  $X$  and every  $p, q, i \geq 0$ , Deligne defined a complex vector space  $H^{p,q;i}(X)$  that is a subquotient of the singular cohomology  $H^i(X; \mathbb{C})$  in complex coefficients. The dimension  $h^{p,q;i}(X)$  of  $H^{p,q;i}(X)$  is called the *mixed Hodge number* of  $X$  of *Hodge type*  $(p, q)$ . We have

$$h^i(X) = \sum_{p,q \geq 0} h^{p,q;i}(X), \quad (\text{VI.1.9})$$

so the data of mixed Hodge numbers refine the data of Betti numbers.

**Theorem VI.1.1** (Main Result). *Let  $X$  be a connected compact smooth complex variety of dimension  $d$  with  $r \geq 1$  points punctured (in particular,  $X$  is never compact). Let  $P$  be a point of  $X$ , then the mixed Hodge numbers of the further-punctured variety  $X - P$  are given by*

$$h^{p,q;i}(\text{Conf}^n(X - P)) = \sum_{t \geq 0} h^{p-dt, q-dt; i-(2d-1)t}(\text{Conf}^{n-t}(X)). \quad (\text{VI.1.10})$$

Equivalently,

$$\begin{aligned} & \sum_{p,q,i,n \geq 0} h^{p,q;i}(\text{Conf}^n(X - P)) x^p y^q (-u)^i t^n \\ &= \frac{1}{1 + (xy)^d u^{2d-1} t} \sum_{p,q,i,n \geq 0} h^{p,q;i}(\text{Conf}^n(X)) x^p y^q (-u)^i t^n. \end{aligned} \quad (\text{VI.1.11})$$

The formula (VI.1.11) describes the relationship between mixed Hodge numbers of  $\text{Conf}^n(X - P)$  for all  $n$  and mixed Hodge numbers of  $\text{Conf}^n(X)$  for all  $n$ . If we substitute  $x = y = 1$  in (VI.1.11), we recover Kallel's formula (VI.1.7) about Betti numbers.

To explain the connection between Theorem VI.1.1 and the point counting formula (VI.1.8), we introduce an analogue of point counting for a complex variety. For a connected smooth complex variety  $X$  of dimension  $d$ , define the *E-polynomial* (following the notation of [HRV08]) as

$$E(X; x, y) = \sum_{p,q,i \geq 0} (-1)^i h^{p,q;i}(X) x^{d-p} y^{d-q} \in \mathbb{Z}[x, y]. \quad (\text{VI.1.12})$$

The notion of *E-polynomial* can be uniquely extended to all complex varieties, such that the followings hold:

- (a) For any complex varieties  $X$  and  $Y$ , we have  $E(X \times Y; x, y) = E(X; x, y)E(Y; x, y)$ ;
- (b) If  $Z$  is a closed subvariety of a complex variety  $X$ , and  $U = X - Z$  is the complement of  $Z$ , then  $E(X; x, y) = E(U; x, y) + E(Z; x, y)$ .

In other words, the *E-polynomial* behaves well with the cartesian product and “cut-and-paste”, just like the point count for varieties over a finite field. Like the *E-polynomial*, an invariant that satisfies properties (a) and (b) above is said to be *motivic*. We remark that Betti numbers and mixed Hodge numbers are not motivic.

Because the *E-polynomial* is motivic, the formula [VW15, Theorem 5.9] of Vakil and

Wood from which the point count formula (VI.1.8) is derived also implies

$$\sum_{n=0}^{\infty} E(\text{Conf}^n(X - P); x, y)t^n = \frac{1}{1+t} \sum_{n=0}^{\infty} E(\text{Conf}^n(X); x, y)t^n, \quad (\text{VI.1.13})$$

where  $X$  is any complex variety and  $P$  is a point of  $X$ .

Given (VI.1.11), if we substitute  $x \mapsto x^{-1}, y \mapsto y^{-1}, u \mapsto 1$  and  $t \mapsto (xy)^{dt}$ , then we recover (VI.1.13) about  $E$ -polynomials. We point out that the notion of  $E$ -polynomial of a complex variety is indeed a reasonable analogue of the point count of a variety over  $\mathbb{F}_q$ : in fact, due to a result of Nicholas Katz [HRV08, §6], the  $E$ -polynomial of a complex variety  $X$  is uniquely determined by the point count of spreadings out of  $X$  to “enough” finite fields.

In conclusion, for a complex variety satisfying the assumption of Theorem VI.1.1, the formula (VI.1.11) is a common refinement of Kallel’s formula (VI.1.7) about Betti numbers and Vakil and Wood’s formula (VI.1.13) about  $E$ -polynomials, which can be viewed as an analogue of the point count formula (VI.1.8) but for complex varieties.

### VI.1.3: Applications to algebraic curves

Let  $\Sigma_{g,r}$  be a smooth projective algebraic curve of genus  $g$  minus  $r$  points. Drummond-Cole and Knudsen [DCK17] computed the Betti numbers  $h^i(\text{Conf}^n(\Sigma_{g,r}))$ . As an application of Theorem VI.1.1, we describe the picture about the mixed Hodge numbers  $h^{p,q;i}(\text{Conf}^n(\Sigma_{g,r}))$ . The nicest case is when  $g = 0, r \geq 1$ , where the rational mixed Hodge structure on  $H^i(\text{Conf}^n(\Sigma_{0,r}))$  was shown to be pure of weight  $2i$  by Kim [Kim94]. When  $g = r = 1$ , Cheong and the author [CH22] proved that the rational mixed Hodge structure on  $H^i(\text{Conf}^n(\Sigma_{1,1}))$  is pure of weight  $w(i) = \lfloor 3i/2 \rfloor$ . In both cases above ( $g = 0, r \geq 1$  or  $g = r = 1$ ), thanks to the purity statement, the motivic formula of Vakil and Wood gives rise to a rational generating function in four variables that encodes the mixed Hodge numbers  $h^{p,q;i}(\text{Conf}^n(\Sigma_{g,r}))$  with  $p, q, i, n$  varying, as is explained in [CH22]. The methods of [CH22] can also be applied to the case  $g \geq 2, r = 1$  to give an explicit basis for  $H^{p,q;i}(\text{Conf}^n(\Sigma_{g,1}))$ , thus giving the mixed Hodge numbers as a counting formula, except that the mixed Hodge structure on  $H^i(\text{Conf}^n(\Sigma_{g,1}))$  is no longer pure, so that the “motivic trick” does not work to give a rational generating function.

Applying Theorem VI.1.1 to the formula in [CH22] for  $g = r = 1$ , we obtain the mixed Hodge numbers  $h^{p,q;i}(\text{Conf}^n(\Sigma_{g,r}))$  for  $g = 1, r = 2$ . Similarly we can obtain an analogous formula for  $g = 1$  and any  $r \geq 2$  by applying Theorem VI.1.1 repetitively. In particular, we may generate an explicit table of mixed Hodge numbers of  $\text{Conf}^n(\Sigma_{1,r})$  for  $r \geq 2$ . We emphasize that this very table would show that the mixed Hodge structure on  $H^i(\text{Conf}^n(\Sigma_{1,r}))$  is not

pure for all  $r \geq 2$  (except possibly for small  $i$  and  $n$ ), so the computation of  $h^{p,q;i}(\text{Conf}^n(\Sigma_{1,r}))$  for  $r \geq 2$  could not be a consequence of the “motivic trick”. For  $g \geq 2, r \geq 1$ , it turns out that the mixed Hodge structure on  $H^i(\text{Conf}^n(\Sigma_{g,r}))$  is not pure, either, but Theorem VI.1.1 asserts that the mixed Hodge numbers of  $\text{Conf}^n(\Sigma_{g,r})$  (for all  $n$ ) are determined by the mixed Hodge numbers of  $\text{Conf}^n(\Sigma_{g,1})$  (for all  $n$ ).

Theorem VI.1.1 does not work to relate  $\text{Conf}^n(\Sigma_{g,0})$  and  $\text{Conf}^n(\Sigma_{g,1})$ . Fortunately, when  $r = 0$ , Pagaria [Pag] obtained a (not necessarily rational) generating function that computes all mixed Hodge numbers  $h^{p,q;i}(\text{Conf}^n(\Sigma_{g,0}))$ . In conclusion, the formulas for  $h^{p,q;i}(\text{Conf}^n(\Sigma_{g,r}))$  can be organized into three essentially different cases: the compact case  $r = 0$ , following [Pag]; the one-punctured case  $r = 1$ , following [CH22]; and the multi-punctured case  $r \geq 2$ , which can be directly obtained from the  $r = 1$  case via Theorem VI.1.1.

#### VI.1.4: Generalizations and refinements

The conclusion of Theorem VI.1.1 and Kallel’s Betti number formula (VI.1.7) generalize to more examples of  $X$ . For the purpose of this paper, we only consider mixed Hodge structures in rational coefficients. Given a rational number  $\lambda \geq 0$ , we say a complex variety  $X$  to be *pure of slope*  $\lambda$  if the mixed Hodge structure of  $H^i(X, \mathbb{Q})$  is pure of weight  $\lambda \cdot i$  for any integer  $i \geq 0$ , namely, the mixed Hodge number  $h^{p,q;i}(X)$  is zero unless  $p + q = \lambda \cdot i$ . We point out that part of the requirement is that  $H^i(X, \mathbb{Q}) = 0$  for all  $i$  such that  $\lambda \cdot i$  is not an integer. By [Del75, §7, p. 82], the slope of a smooth complex variety (if exists) must be a rational number  $1 \leq \lambda \leq 2$ .

*Remark VI.1.2.* Technically, the slope may not be unique. If  $h^i(X) = 0$  for all  $i > 0$ , then a slope exists and is arbitrary; in any other cases, the slope is unique (if exists). For the purpose of convenience only, we say *the slope of  $X$  is 1* if  $h^i(X) = 0$  for all  $i > 0$ .

**Theorem VI.1.3.** *Let  $\overline{X}$  be a connected smooth noncompact complex variety that is pure of slope  $\lambda$  for some rational number  $1 \leq \lambda \leq 2$ . Let  $X$  be an  $r$ -puncture of  $\overline{X}$  for some  $r \geq 0$ ; in other words,  $X$  is of the form  $X = \overline{X} - \{P_1, \dots, P_r\}$  where  $P_i$  are points of  $\overline{X}$ . Then for any point  $P$  of  $X$ , the conclusion of Theorem VI.1.1 holds as well. As a consequence, Kallel’s (VI.1.7) holds for  $X$  and  $P$ .*

Theorem VI.1.1 is a special case of Theorem VI.1.3 because for any multi-punctured variety  $X$  in the setting of Theorem VI.1.1, we can choose  $\overline{X}$  in Theorem VI.1.3 to be a one-punctured variety (a connected smooth compact variety minus one point). It was noted by Dupont [Dup16, Theorem 2.10] that a one-punctured variety is pure of slope 1, so Theorem

VI.1.3 applies. Therefore, the proof of Theorem VI.1.1 is complete after we prove Theorem VI.1.3.

To list a few other examples of varieties that satisfy the assumption for  $\overline{X}$  (see Section VI.5 for details),

- $\overline{X}$  is an affine space  $\mathbb{C}^d$ , or the complement of a hyperplane arrangement therein. In this case,  $\overline{X}$  is pure of slope 2.
- $\overline{X}$  is the torus  $(\mathbb{C}^*)^d$ , or the complement of a union of 1-codimensional subtori therein. In this case,  $\overline{X}$  is pure of slope 2.
- $\overline{X}$  is the complement of a smooth plane curve in the projective plane  $\mathbb{P}^2$ . In this case,  $X$  is pure of slope 3/2.
- If  $\overline{X}$  is pure of slope  $\lambda$ , then so is a smooth quotient of  $\overline{X}$  by a finite group. For example, we could take  $\overline{X} = \text{Conf}^m(\mathbb{C})$ , in which case  $\overline{X}$  is pure of slope 2.

By taking  $\overline{X}$  from the list above, and possibly puncturing finitely many points, we get abundant examples of even-dimensional noncompact manifolds where Kallel's formula (VI.1.7) holds. We remark that our current proof of the purely topological statement (VI.1.7) for these manifolds requires the structure of complex variety and mixed Hodge theory.

The proof of Theorem VI.1.3 depends on an equivariant result, which is itself a strong refinement of Theorem VI.1.3. The isomorphism in the next theorem is in the category of split mixed Hodge structures, a category that encodes the complex vector spaces  $H^{p,q}(M)$  associated to a mixed Hodge structure  $M$  (and thus the mixed Hodge numbers  $h^{p,q}(M) := \dim_{\mathbb{C}} H^{p,q}(M)$ ) but forgets the extra structures encoded in the weight filtration and the Hodge filtration. A *split mixed Hodge structure* (over  $\mathbb{Q}$ ) is a direct sum of pure Hodge structures. An  $S_n$ -*split mixed Hodge structure* is a split mixed Hodge structure together with an action by  $S_n$ . For a mixed Hodge structure  $M$ , its associated graded  $\text{gr } M$  with respect to the weight filtration is a split mixed Hodge structure.

**Theorem VI.1.4.** *Let  $X$  be a complex variety as in Theorem VI.1.3, and  $P$  be a point of  $X$ . Recall that  $F(X, n)$  denotes the  $n$ -th ordered configuration space of  $X$ . Then we have a noncanonical isomorphism of  $S_n$ -representations*

$$H^i(F(X - P, n), \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \text{Ind}_{S_{n-t}}^{S_n} H^{i-(2d-1)t}(F(X, n-t), \mathbb{Q}). \quad (\text{VI.1.14})$$

Moreover, we have a noncanonical isomorphism in the (semi-simple) category of  $S_n$ -split

*mixed Hodge structures*

$$\mathrm{gr} H^i(F(X - P, n), \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} \mathrm{gr} H^{i-(2d-1)t}(F(X, n-t), \mathbb{Q})(-d \cdot t), \quad (\text{VI.1.15})$$

where  $(-d \cdot t)$  denotes the Tate twist.

Here, the operator  $\mathrm{Ind}_{S_{n-t}}^{S_n}$  denotes the induction of an  $S_{n-t}$ -representation to an  $S_n$ -representation, where  $S_{n-t}$  is the subgroup of  $S_n$  consisting of permutations that permute the first  $n-t$  elements and fix the last  $t$  elements. The Tate twist shifts a mixed Hodge structure according to the rule

$$H^{p,q}(M(n)) = H^{p+n,q+n}(M). \quad (\text{VI.1.16})$$

As a special case of Shapiro's lemma [Wei94, p. 172] (or alternatively, the Frobenius reciprocity), we have the following standard fact: for any subgroup  $H$  of a finite group  $G$ , and a representation  $V$  of  $H$ , then the  $G$ -invariant of the induction of  $V$  is isomorphic to the  $H$ -invariant of  $V$  as a vector space:

$$(\mathrm{Ind}_H^G V)^G \cong V^H. \quad (\text{VI.1.17})$$

As a result, by taking the  $S_n$ -invariants of both sides of (VI.1.15) and extracting the mixed Hodge numbers of Hodge type  $(p, q)$ , we get

$$h^{p,q;i}(\mathrm{Conf}^n(X - P)) = \sum_{t=0}^{\infty} h^{p-dt,q-dt;i-(2d-1)t}(\mathrm{Conf}^{n-t}(X)), \quad (\text{VI.1.18})$$

which is equivalent to our main result (VI.1.11).

We conclude this section with an alternative form of (VI.1.15) that notably does not involve the dimension  $d$ . Let  $H_c^i(X; \mathbb{Q})$  denote the compactly supported cohomology of  $X$ , which is also equipped with a mixed Hodge structure. Using the Poincaré duality, the formula (VI.1.15) is equivalent to the following isomorphism of  $S_n$ -split mixed Hodge structures:

$$\mathrm{gr} H_c^i(F(X - P, n), \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} \mathrm{gr} H_c^{i-t}(F(X, n-t), \mathbb{Q}). \quad (\text{VI.1.19})$$

### VI.1.5: Overview of methods

Our approach to Theorem VI.1.1 is to prove an equivariant theorem (Theorem VI.1.4) about the cohomology of ordered configuration spaces  $F(X, n)$ . To prove Theorem VI.1.4, we



use a spectral sequence similar to a Leray spectral sequence described by Totaro [Tot96]. The spectral sequence we use has its  $E_1$  page as a differential graded algebra with explicit generators and relations, and the spectral sequence degenerates on the  $E_2$  page; these allow explicit computation of cohomology groups of  $F(X, n)$ . We point out that in order to have degeneration on  $E_2$ , we must use a spectral sequence different from the one described in [Tot96].

Like the Leray spectral sequence in [Tot96], our spectral sequence also remembers mixed-Hodge-theoretic information of  $F(X, n)$  and respects the action of  $S_n$  on  $F(X, n)$ . Looking at a similar spectral sequence for  $F(X - P, n)$  and comparing it with the spectral sequence for  $F(X, n)$ , we are able to construct (artificially using the generators and relations on the  $E_1$  page) an equivariant isomorphism that expresses the cohomology of  $F(X - P, n)$  in terms of the cohomology of  $F(X, m)$  for all  $m \leq n$ ; this is Theorem VI.1.4. Taking the  $S_n$ -invariant parts of both sides, we get an identity that compares the mixed Hodge numbers of  $\text{Conf}^n(X)$  (for all  $n$ ) and  $\text{Conf}^n(X - P)$  (for all  $n$ ), which is the identity required in Theorem VI.1.1.

We now state the desired properties of the spectral sequence we use.

**Theorem VI.1.5.** *Let  $\overline{X}$  be a connected smooth complex variety and  $X$  be an  $r$ -puncture of  $\overline{X}$ , where  $r \geq 0$ . Then for any point  $P$  of  $X$ , the following statements hold:*

- (a) *There exist spectral sequences of  $S_n$ -mixed Hodge structures  $E_1^{i,j}(X - P, n) \implies H^{i-j}(F(X - P, n); \mathbb{Q})$  and  $E_1^{i,j}(X, n) \implies H^{i-j}(F(X, n); \mathbb{Q})$  with explicit description of the first page (see Section VI.3) for all  $n$ , such that there exists an isomorphism (see Lemma VI.3.1) of  $S_n$ -mixed Hodge structures*

$$\Phi : E_1^{i,j}(X, n) \oplus \text{Ind}_{S_{n-1}}^{S_n} E_1^{i-2d, j-1}(X - P, n - 1)(-d) \rightarrow E_1^{i,j}(X - P, n), \quad (\text{VI.1.20})$$

where  $(-d)$  denotes the Tate twist. The construction of both sequences  $E(X - P, n)$  and  $E(X, n)$  depend on  $\overline{X}$  despite the notation.

- (b) *If  $\overline{X}$  is noncompact, then  $\Phi$  commutes with the first-page differential map  $d_1^{i,j} : E_1^{i,j} \rightarrow E_1^{i, j-1}$ .*
- (c) *If  $\overline{X}$  is pure of slope  $\lambda$ , then both spectral sequences  $E(X - P, n)$  and  $E(X, n)$  degenerate on  $E_2$ , namely, all higher-page differentials  $d_r$  ( $r \geq 2$ ) vanish.*

We point out that the important assumptions of Theorem VI.1.3 about  $\overline{X}$  result from the assumptions of parts (b) and (c) of Theorem VI.1.5.

## VI.1.6: Organization of the paper

The rest of the paper will be devoted to the proof of Theorem VI.1.4. In Section VI.2, we describe a general spectral sequence (Proposition VI.2.3) used in the construction of  $E(X - P, n)$  and  $E(X, n)$  in Theorem VI.1.5. In Section VI.3, we construct the isomorphism  $\Phi$  in Theorem VI.1.5 explicitly, and prove each part of Theorem VI.1.5 using explicit computations. In Section VI.4, we prove Theorem VI.1.4 from Theorem VI.1.5, and thus conclude the proof of all of the main results. In Section VI.5, we discuss the possibilities and difficulties of further generalizing Theorem VI.1.3.

## VI.2: Description of the spectral sequences

### VI.2.1: Arrangements and Orlik–Solomon algebra

In order to construct the spectral sequences in Theorem VI.1.5, we give an introduction of necessary preliminaries about hyperplane arrangements and the Orlik–Solomon algebra. For a reference, see for instance [OT92].

A *hyperplane arrangement* is a set  $\mathcal{A} := \{Y_1, \dots, Y_h\}$  of complex hyperplanes in  $\mathbb{C}^n$ . A *stratum*  $F$  of  $\mathcal{A}$  is the intersection of zero or more hyperplanes in  $\mathcal{A}$  (so  $\mathbb{C}^n$  is also a stratum). The *lattice*  $L(\mathcal{A})$  of  $\mathcal{A}$  is the partially ordered set (poset) of strata of  $\mathcal{A}$ , ordered by inclusion. It has a top element  $\hat{1} := \mathbb{C}^n$ . In the rest of this section, all of the constructions associated to  $\mathcal{A}$  are determined by the poset  $L(\mathcal{A})$  alone.

The *rank* of a stratum  $F$  is its complex codimension in  $\mathbb{C}^n$ , or equivalently, the length of any maximal chain from  $F$  to  $\hat{1}$ . A subset  $S$  of  $\mathcal{A}$  is called (1) *independent*, if  $\bigcap S := \bigcap_{F \in S} F \neq \emptyset$  and  $\text{rk}(\bigcap S) = |S|$ ; (2) *dependent*, if  $\bigcap S \neq \emptyset$  and  $\text{rk}(\bigcap S) < |S|$ ; (3) *vanishing*, if  $\bigcap S = \emptyset$ .

Define the algebra  $B(\mathcal{A})$  to be the free graded-commutative  $\mathbb{Z}$ -algebra (i.e. the exterior algebra) generated by degree-one elements  $e_Y, Y \in \mathcal{A}$ . Let  $\partial : B(\mathcal{A}) \rightarrow B(\mathcal{A})$  be the unique  $\mathbb{Z}$ -linear map on  $B(\mathcal{A})$  that satisfies the graded Leibniz rule for derivations and  $\partial e_Y = 1$  for  $Y \in \mathcal{A}$ . It has the property that  $\partial^2 = 0$ . For an ordered subset  $S$  of  $\mathcal{A}$ , we denote  $e_S := \prod_{Y \in S} e_Y$ , with  $e_\emptyset = 1$  and  $\bigcap \emptyset = X$  by convention. There is an important observation ([OT92, Lemma 3.7]) for any nonempty  $S$  that  $e_S$  is always divisible by  $\partial(e_S)$  in  $A(\mathcal{A})$ . Define  $I(\mathcal{A})$  to be the ideal of  $B(\mathcal{A})$  generated by  $e_S$  for  $S$  vanishing and  $\partial(e_S)$  for  $S$  dependent. It turns out (as a consequence of the important observation above) that  $I(\mathcal{A})$  is generated by  $e_S$  for minimal vanishing sets  $S$  and  $\partial(e_S)$  for minimal dependent sets  $S$ . Define the *Orlik–Solomon algebra* of  $\mathcal{A}$  (or of  $L(\mathcal{A})$ ) as  $A(\mathcal{A}) := B(\mathcal{A})/I(\mathcal{A})$ . Denote by  $g_Y$  the image of  $e_Y \in B(\mathcal{A})$  in  $A(\mathcal{A})$  under the quotient map. The differential  $\partial$  on  $B(\mathcal{A})$

descends to a differential  $\partial : A(\mathcal{A}) \rightarrow A(\mathcal{A})$ , making  $A(\mathcal{A})$  a differential graded algebra.

More concretely,  $A(\mathcal{A})$  is the graded-commutative  $\mathbb{Z}$ -algebra generated by degree-one elements  $g_Y$ ,  $Y \in \mathcal{A}$  with the following relations:

- (a)  $g_S = 0$  if  $S$  is vanishing;
- (b) If  $S = \{Y_{i_1}, \dots, Y_{i_k}\}$  is dependent, then

$$\sum_{j=1}^k (-1)^{j+1} g_{Y_{i_1}} \dots \widehat{g_{Y_{i_j}}} \dots g_{Y_{i_k}} = 0 \quad (\text{VI.2.1})$$

where the notation  $\widehat{g_{Y_{i_j}}}$  means skipping  $j$ -th factor in the product. Note that relation (b) implies that  $g_S = 0$  for  $S$  dependent (again due to the observation [OT92, Lemma 3.7]).

The *complement* of  $\mathcal{A}$  is the complex variety  $M(\mathcal{A}) := \mathbb{A}^n - \bigcup_{Y \in \mathcal{A}} Y$ . The cohomology ring  $H^*(M(\mathcal{A})) := \bigoplus_{i=0}^{\infty} H^i(M(\mathcal{A}), \mathbb{Z})$  is isomorphic to the graded algebra  $A(\mathcal{A})$ .

Let  $F$  be a stratum of  $\mathcal{A}$ . Consider the subarrangement  $\mathcal{A}_F := \{Y \in \mathcal{A} : Y \supseteq F\}$ , then we have  $M(\mathcal{A}) \subseteq M(\mathcal{A}_F)$ , and the pullback of the inclusion map gives a morphism of graded algebras  $H^*(M(\mathcal{A}_F)) \rightarrow H^*(M(\mathcal{A}))$ . Via the identification above, this is the ring homomorphism  $A(\mathcal{A}_F) \rightarrow A(\mathcal{A})$  that sends  $g_Y$  to  $g_Y$ . If  $F' \subseteq F$  are two strata of  $\mathcal{A}$ , then we have a natural map  $A(\mathcal{A}_F) \rightarrow A(\mathcal{A}_{F'})$ .

Let  $A_F(\mathcal{A})$  be the abelian subgroup of  $A(\mathcal{A})$  generated by  $g_S$  with  $\bigcap S = F$ . Then  $A(\mathcal{A}) = \bigoplus_F A_F(\mathcal{A})$ , where  $F$  ranges over all strata of  $\mathcal{A}$ . Also,  $A_F(\mathcal{A})$  is the image of  $A(\mathcal{A}_F)_{\text{rk } F} \rightarrow A(\mathcal{A})_{\text{rk } F}$ , where  $(\cdot)_d$  is taking the degree- $d$  part.

### VI.2.2: Lattice spectral sequence

Consider a smooth complex variety  $V$  and a collection of smooth  $d$ -codimensional closed subvarieties  $\mathcal{A} = \{Y_1, \dots, Y_h\}$ . We say that  $Y_1, \dots, Y_h$  intersect *like a hyperplane arrangement* if

- (a) For any  $S \subseteq \mathcal{A}$ , the intersection  $\bigcap S$  (if nonempty) is smooth and connected. Call such an intersection a *stratum* of  $\mathcal{A}$ .
- (b) Every stratum has codimension a multiple of  $d$ .
- (c) The poset of strata of  $\mathcal{A}$  is isomorphic to the lattice of some hyperplane arrangement of a complex vector space. Denote this poset by  $L(\mathcal{A})$ , also called the *lattice* of  $\mathcal{A}$ .

- (d) The set  $\mathcal{A}$  is precisely the set of rank one strata in  $L(\mathcal{A})$ , and every stratum  $F \in L(\mathcal{A})$  satisfies  $\text{codim}(F) = d \cdot \text{rk}(F)$ . (Recall the *rank* of a stratum  $F$  is the length of any maximal chain from  $F$  to  $\hat{1} = V$ .)

(The last two statements are actually redundant as they are consequences of the first two.)

We define  $A(\mathcal{A})$  to be the Orlik–Solomon algebra associated to the lattice  $L(\mathcal{A})$ . Denote  $M(\mathcal{A}) := V - \bigcup_{i=1}^h Y_i$ .

Tosteson [Tos16] describes a spectral sequence converging to  $H^*(M(\mathcal{A}), \mathbb{Z})$  that works in a general setting. In the special case we describe above, the spectral sequence can be vastly simplified into a form similar to [Dup16, Theorem 3.1].

**Proposition VI.2.1.** *Let  $V$  and  $\mathcal{A}$  be as above. Then there is a spectral sequence of integral-coefficient mixed Hodge structures  $E_1^{i,j}(\mathcal{A}) \implies H^{i-j}(M(\mathcal{A}), \mathbb{Z})$  such that the abelian group  $E_1(\mathcal{A}) := \bigoplus_{i,j \geq 0} E_1^{i,j}(\mathcal{A})$  bigraded by  $(i, j)$  is given by*

$$E_1(\mathcal{A}) = \bigoplus_{F \in L(\mathcal{A})} H^*(F, \mathbb{Z}) \otimes A_F(\mathcal{A}) \quad (\text{VI.2.2})$$

where  $H^i(F, \mathbb{Z})$  has bidegree  $(i, 0)$  and  $A_F(\mathcal{A})$  has bidegree  $(2 \text{codim}_{\mathbb{C}} F, \text{rk } F)$  and Hodge type  $(\text{codim}_{\mathbb{C}} F, \text{codim}_{\mathbb{C}} F)$ .

The group  $E_1(\mathcal{A})$  has a structure of graded-commutative algebra<sup>1</sup> induced from the algebra structure of  $A(\mathcal{A}) = \bigoplus_F A_F(\mathcal{A})$  and the cup product on  $H^*(F, \mathbb{Z})$ .

The first-page differential map  $d_1^{i,j} : E_1^{i,j} \rightarrow E_1^{i,j-1}$  is given by the Gysin map on the cohomology and the differential  $\partial$  on  $A(\mathcal{A})$ . It makes  $E_1(\mathcal{A})$  a differential graded algebra<sup>2</sup>.

The spectral sequence is functorial in automorphisms of  $V$  that preserve  $\mathcal{A}$ .

*Remark VI.2.2.* In the case where  $\mathcal{A}$  is the big diagonal arrangement that gives the ordered configuration space  $F(X, n)$  of a  $d$ -dimensional variety  $X$ , the spectral sequence here is the same as [Tot96] but with a degree shifting, so that  $E_1^{2d,1}$  here corresponds to Totaro’s  $E_{2d}^{0,2d-1}$ . The spectral sequence in Proposition VI.2.1 can also be viewed as a high-codimensional analogue of the Orlik–Solomon spectral sequence in [Loo93, §2] and [Dup15].

*Proof.* One can reuse Totaro’s argument in [Tot96] based on the result [GM88, pp. 237–239] of Goresky and MacPherson about arrangements of  $k$ -codimensional subspaces of  $\mathbb{R}^n$  whose all strata have codimension a multiple of  $k$  (see the Remark after the proof of Lemma 3

<sup>1</sup>The direct summand  $E_1^{i,j}$  is assigned the degree  $i - j$ .

<sup>2</sup>A *differential graded algebra* (dga) is a graded-commutative algebra equipped with a linear map  $d$  of degree 1 (namely, sending a degree- $k$  element to a degree- $(k + 1)$  element), called the differential map, such that  $d \circ d = 0$  and the graded Leibniz rule is satisfied.

in [Tot96] for a discussion). This recognizes  $E_1(\mathcal{A})$  as the  $E_{2d}$  page of the Leray spectral sequence of  $F(X_r, n) \hookrightarrow X^n$ , where a dga structure is present. See also [MP20, Lemma 3.1].

Alternatively, one can use the spectral sequence described in [Tos16, Theorem 1.8]:

$$E_1^{i,j}(\mathcal{A}) = \bigoplus_{F \in L(\mathcal{A})} \tilde{H}_{j-2}((F, \hat{1}); H^i(V, V - F; \mathbb{Z})) \quad (\text{VI.2.3})$$

where  $\tilde{H}_{j-2}((F, \hat{1}))$  is the reduced homology of the order complex of the poset  $(F, \hat{1})$ , with a special convention when  $F = \hat{1}$ . We refer the reader to [Wac07] to a detailed account for these concepts.

Since  $L(\mathcal{A})$  is isomorphic to the lattice of a hyperplane arrangement, we have

$$\tilde{H}_{j-2}((F, \hat{1}); \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(F, \hat{1})|} = A_F(\mathcal{A}), & j = \text{rk } F \\ 0 & j \neq \text{rk } F \end{cases} \quad (\text{VI.2.4})$$

for all  $F \in L(\mathcal{A})$  (including  $F = \hat{1}$ ), where  $\mu$  is the Möbius function of the lattice  $L(\mathcal{A})$  (see [Fol66, Theorem 4.1] and [OT92, §4.5]).

By the tubular neighborhood theorem and the excision theorem, we have  $H^i(V, V - F; \mathbb{Z}) \cong H^i(\mathcal{N}_F, \mathcal{N}_F - F; \mathbb{Z})$ , where  $\mathcal{N}_F$  is the normal bundle of  $F$  in  $V$ . Since we are considering complex manifolds, the normal bundle has a canonical orientation, which gives a canonical Thom isomorphism

$$H^i(V, V - F; \mathbb{Z}) \cong H^{i-2 \text{codim}_{\mathbb{C}} F}(F, \mathbb{Z}). \quad (\text{VI.2.5})$$

Combining the above, we get  $E_1^{i,j}(\mathcal{A}) = \bigoplus_{\text{rk } F=j} H^{i-2 \text{codim}_{\mathbb{C}} F}(F, \mathbb{Z}) \otimes A_F(\mathcal{A})$  as required. For the compatibility with the mixed Hodge structure, see [Pet17, §3.2]. The structure of differential graded algebra on  $E_1(\mathcal{A})$  can be constructed using the functoriality of Tosteson's spectral sequence along the diagonal map  $V \rightarrow V \times V$ .  $\square$

### VI.2.3: Arrangements arising from punctured varieties

In this section, we provide a simplified description of the spectral sequence in the following special setting. Consider a connected smooth complex variety  $X$  of dimension  $d$  and distinct points  $P^1, \dots, P^r$  ( $r \geq 1$ ) of  $X$ . (We use superscripts for the points for future convenience.) Fix  $n \geq 0$ , and consider the arrangement  $\mathcal{A}$  of  $X^n$  consisting of the following  $d$ -codimensional

closed subvarieties:

$$\begin{aligned}\Delta_{ij} &:= \{(x_1, \dots, x_n) \in X^n : x_i = x_j\} \\ \Delta_i^s &:= \{(x_1, \dots, x_n) \in X^n : x_i = P^s\}\end{aligned}$$

for  $1 \leq i \neq j \leq n$  and  $1 \leq s \leq r$ . Then the complement  $M(\mathcal{A})$  of the arrangement  $\mathcal{A}$  is the ordered configuration space  $F(X_r, n)$ , where  $X_r := X - \{P^1, \dots, P^r\}$ .

The goal of this section is Proposition VI.2.3, an explicit description of the differential graded algebra  $E_1(\mathcal{A}) \otimes \mathbb{Q}$  in terms of generators and relations. In the rest of this section, *we assume every cohomology is in rational coefficients.*

We denote by  $p_i : X^n \rightarrow X$  the projection map onto the  $i$ -th coordinate, and denote  $p_i^*(\alpha)$  by  $\alpha_i$ . Since we are working in rational coefficients, by Künneth's formula  $H^*(X^n) \cong H^*(X)^{\otimes n}$ , the cohomology ring  $H^*(X^n)$  is generated by elements of the form  $\alpha_i$  where  $\alpha \in H^*(X)$  and  $1 \leq i \leq n$ . For  $i \neq j$ , let  $p_{ij} : X^n \rightarrow X^2$  be the projection map  $(x_1, \dots, x_n) \mapsto (x_i, x_j)$ . Let  $\Delta$  be the diagonal of  $X^2$ . In the notation of above, we have

$$p_{ij}^{-1}(\Delta) = \Delta_{ij} \tag{VI.2.6}$$

$$p_i^{-1}(P^s) = \Delta_i^s \tag{VI.2.7}$$

To a smooth irreducible  $d$ -codimensional closed subvariety  $Z$  of any smooth variety  $Y$ , we associate a cohomology class  $[Z] \in H^{2d}(Y)$  given by the image of the canonical generator  $1 \in H^0(Z)$  under the Gysin map  $H^*(Z) \rightarrow H^{*+2d}(Y)$ . One way to define the Gysin map is using the Poincaré dual  $H^i(Z) = H_c^{2(n-d)-i}(Z)^\vee \rightarrow H_c^{2n-(i+2d)}(Y)^\vee = H^{i+2d}(Y)$  of the pullback map  $H^{2n-(i+2d)}(Y) \rightarrow H^{2(n-d)-i}(Z)$ . The class of a closed subvariety satisfies the following property: if  $f : X \rightarrow Y$  is a flat map of constant relative dimension, and  $D$  is an algebraic cycle of  $Y$ , then  $f^*[D] = [f^{-1}(D)]$ , where  $f^{-1}(D)$  is the pullback.

Define  $E_1(X_r, n) := E_1(\mathcal{A}) \otimes \mathbb{Q}$ . We point out that the definition of  $E_1(X_r, n)$  depends on  $X$  together with  $\{P^1, \dots, P^r\}$  despite the notation.

**Proposition VI.2.3.** *Let  $\mathcal{A}$  be the arrangement above. Then the differential graded algebra  $E_1(X_r, n)$  is given by*

$$E_1(X_r, n) := \frac{H^*(X^n; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[g_{ij}, g_i^s : 1 \leq i \neq j \leq n, 1 \leq s \leq r]}{(\text{relations})}$$

with  $H^k(X^n)$  having bidegree  $(k, 0)$ , and  $g_{ij}$  and  $g_i^s$  having bidegree  $(2d, 1)$  and Hodge type

$(d, d)$ , subject to relations given by

$$g_{ij} = g_{ji} \tag{VI.2.8}$$

$$g_{ij}g_{jk} + g_{jk}g_{ki} + g_{ki}g_{ij} = 0 \text{ for } i, j, k \text{ distinct} \tag{VI.2.9}$$

$$g_{ij}\alpha_i = g_{ij}\alpha_j \text{ for } \alpha \in H^*(X) \tag{VI.2.10}$$

$$g_i^s \alpha_i = 0 \text{ for } \alpha \in H^{\geq 1}(X) := \bigoplus_{p \geq 1} H^p(X) \tag{VI.2.11}$$

$$g_i^s g_j^s - g_{ij} g_i^s + g_{ij} g_j^s = 0 \tag{VI.2.12}$$

$$g_i^s g_i^t = 0 \text{ for } s \neq t \tag{VI.2.13}$$

The differential  $d := d_1$  on the  $E_1$  page is determined by

$$dg_{ij} = p_{ij}^*[\Delta] \tag{VI.2.14}$$

$$dg_i^s = [\Delta_i^s] \tag{VI.2.15}$$

$$d|_{H^*(X^n)} = 0 \tag{VI.2.16}$$

The symmetric group  $S_n$  acts on  $E_1(X_r, n)$  by permuting subscripts.

We note that the notation  $\mathbb{Q}[g_{ij}, g_i^s]$  above does not denote a polynomial ring, but an exterior algebra, since the generators have degree 1.

#### VI.2.4: Proof of Proposition VI.2.3

The statement and the proof idea of Proposition VI.2.3 is similar to Bibby's [Bib16, Theorem 4.1], but we present a proof in full detail here due to the lack of a direct reference.

A stratum  $F$  of  $\mathcal{A}$  can be uniquely indexed by a pair  $(\chi, \sim)$  of a *coloring* function  $\chi : \{1, \dots, n\} \rightarrow \{0, \dots, r\}$  and an equivalence relation  $\sim$  on  $\chi^{-1}(0)$ , according to the rule

$$F_{(\chi, \sim)} = \{(x_1, \dots, x_n) \in X^n : x_i = P^{\chi(i)} \text{ if } \chi(i) \neq 0, \tag{VI.2.17}$$

$$\text{and } x_i = x_j \text{ if } \chi(i) = \chi(j) = 0 \text{ and } i \sim j. \} \tag{VI.2.18}$$

In other words, a coordinate that is colored  $s$  ( $1 \leq s \leq r$ ) is required to take  $P^s$  as value, and the coordinates colored 0 have no such a requirement, but they must agree if they belong to the same block of the partition given by  $\sim$ . It is helpful to think of color 0 as “uncolored”.

The arrangement  $\mathcal{A}$  satisfies the following key property. Recall that every cohomology is in rational coefficients unless otherwise notated.

**Lemma VI.2.4.** *Let  $F = F_{(\chi, \sim)}$  be a stratum of  $\mathcal{A}$  described above. Then the pullback map  $H^*(X^n) \rightarrow H^*(F)$  is surjective. Moreover, its kernel is the ideal generated by  $p_i^* \alpha$  for  $\alpha \in H^{\geq 1}(X)$ ,  $\chi(i) \neq 0$ , and  $p_i^* \alpha - p_j^* \alpha$  for  $\alpha \in H^*(X)$ ,  $\chi(i) = \chi(j) = 0$ ,  $i \sim j$ .*

*Proof.* By Künneth's formula, we can deal with each  $i$  with  $\chi(i) \neq 0$  and each equivalence class of  $\chi^{-1}(0)$  separately. It suffices to prove the lemma for the following cases:

- (a)  $n = 1$ ,  $F = \Delta_1^s$ , which is the point  $P^s$ . Then since  $H^*(F) = H^0(F)$  and both  $X$  and  $F$  are connected, the kernel of  $H^*(X) \rightarrow H^*(F)$  is  $H^{\geq 1}(X)$ .
- (b)  $n \geq 2$ ,  $F = \{(x_1, \dots, x_n) : x_1 = \dots = x_n\}$ . Then  $F \hookrightarrow X^n$  is isomorphic to the diagonal map  $X \hookrightarrow X^n$ ,  $x \mapsto (x, \dots, x)$ , so the kernel of  $H^*(X^n) \rightarrow H^*(F)$  is the same as the kernel of the multiplication map  $H^*(X)^{\otimes n} \rightarrow H^*(X)$ .

We recall a standard fact in commutative algebra, but now we state and prove an extension in the following setting. Let  $R$  be a commutative ring and  $A$  a graded-commutative  $R$ -algebra. The  $R$ -module  $A^{\otimes n} := A \otimes_R \dots \otimes_R A$  has a structure of graded-commutative algebra that satisfies

- For  $1 \leq i \leq n$ , we have a degree-preserving algebra homomorphism  $\theta_i : A \rightarrow A^{\otimes n}$  sending  $a \in A$  to  $1 \otimes \dots \otimes a \otimes \dots \otimes 1$  where  $a$  appears at the  $i$ -th factor;
- $a_1 \otimes \dots \otimes a_n = \theta_1(a_1) \dots \theta_n(a_n)$ .

We claim that the kernel of the multiplication map  $\mu_n : A^{\otimes n} \rightarrow A$ ,  $a_1 \otimes \dots \otimes a_n \mapsto a_1 \dots a_n$  is generated by  $\theta_i(a) - \theta_j(a)$  for all  $1 \leq i, j \leq n$  and  $a \in A$ . Lemma VI.2.4 then follows from the claim by letting  $A = H^*(X)$  and  $R = \mathbb{Q}$ .

It is obvious that  $\theta_i(a) - \theta_j(a)$  is in the kernel. We shall prove the reverse inclusion by induction on  $n$ .

- (i) Case  $n = 2$ . An element of  $\ker \mu_2$  is of the form

$$\sum_{j=1}^h a_j \otimes b_j = \sum_{j=1}^h \theta_1(a_j) \theta_2(b_j)$$

such that  $\sum a_j b_j = 0$  in  $A$ .

Then

$$\sum a_j \otimes b_j = \sum \theta_1(a_j) \theta_2(b_j) - \theta_1\left(\sum a_j b_j\right) \quad (\text{VI.2.19})$$

$$= \sum (\theta_1(a_j) \theta_2(b_j) - \theta_1(a_j) \theta_1(b_j)) \quad (\text{VI.2.20})$$

$$= \sum \theta_1(a_j) (\theta_2(b_j) - \theta_1(b_j)) \quad (\text{VI.2.21})$$



is in the ideal generated by  $\theta_2(b_j) - \theta_1(b_j)$ .

(ii) Case  $n > 2$ . Decompose  $\mu_n$  into two maps:

$$A^{\otimes(n-1)} \otimes_R A \xrightarrow{\mu_{n-1} \otimes 1} A \otimes_R A \xrightarrow{\mu_2} A \quad (\text{VI.2.22})$$

If  $x \in \ker(\mu_n)$ , then  $(\mu_{n-1} \otimes 1)(x)$  must be in the kernel of  $\mu_2$ . By the  $n = 2$  case above, the kernel of  $\mu_n$  is generated by preimages of  $1 \otimes a - a \otimes 1, a \in A$  under  $\mu_{n-1} \otimes 1$ . One of its preimages is  $\theta_n(a) - \theta_{n-1}(a)$ , and all other preimages must differ from this one by an element of  $\ker(\mu_{n-1} \otimes 1) = \ker(\mu_{n-1})\theta_n(A)$ . By the induction hypothesis, the kernel of  $\mu_n$  is contained in the ideal generated by  $\theta_n(a) - \theta_{n-1}(a)$  and  $\theta_i(a) - \theta_j(a), 1 \leq i, j \leq n-1, a \in A$ .

□

Lemma VI.2.4 allows a simplification of Proposition VI.2.1.

**Lemma VI.2.5.** *Let  $\mathcal{A}$  be an arrangement of  $d$ -codimensional subvarieties of a smooth variety  $V$  that intersect like a hyperplane arrangement. Assume in addition that for any stratum  $F$  of  $\mathcal{A}$ , the pullback map  $H^*(V) \rightarrow H^*(F)$  is surjective with kernel  $I_F$ . Then*

$$E_1(\mathcal{A}) \otimes \mathbb{Q} \cong \frac{H^*(V) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F \in L(\mathcal{A}))} \quad (\text{VI.2.23})$$

*Proof.* The graded algebra  $E_1(\mathcal{A}) \otimes \mathbb{Q}$  in Proposition VI.2.1 is given by

$$\begin{aligned} \bigoplus_F H^*(F) \otimes A_F(\mathcal{A}) &= \bigoplus_F \frac{H^*(V)}{I_F} \otimes A_F(\mathcal{A}) \\ &= \frac{\bigoplus_F H^*(V) \otimes A_F(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \\ &= \frac{H^*(V) \otimes \bigoplus_F A_F(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \\ &= \frac{H^*(V) \otimes A(\mathcal{A})}{\sum_F I_F \cdot A_F(\mathcal{A})} \end{aligned}$$

□

We are now ready to compute  $E_1(X_r, n)$ . We need the following technical lemma.

**Lemma VI.2.6.** *Let  $\mathcal{A}$  be a hyperplane arrangement. Let  $S_1, S_2 \subseteq \mathcal{A}$  be disjoint subsets and  $Y \in \mathcal{A}$  be an element not in  $S_1 \sqcup S_2$ . Then inside the algebra  $B(\mathcal{A})$ , we have*

$$\partial_{e_{S_1 \sqcup S_2}} \in (\partial_{e_{S_1 \sqcup \{Y\}}}, \partial_{e_{S_2 \sqcup \{Y\}}})B(\mathcal{A})$$

*Proof.* Write  $A = e_{S_1}$ ,  $B = e_{S_2}$  and  $e = e_Y$ . We have

$$\partial(e\partial(AB)) = (\partial e)(\partial(AB)) - e\partial^2(AB) \quad (\text{VI.2.24})$$

$$= 1 \cdot \partial(AB) - 0 \quad (\text{VI.2.25})$$

$$= \partial(AB) \quad (\text{VI.2.26})$$

Thus

$$\partial(AB) = \partial(e\partial(AB)) \quad (\text{VI.2.27})$$

$$= \partial(e((\partial A)B \pm A\partial B)) \quad (\text{VI.2.28})$$

$$= \pm\partial(eB\partial A) \pm \partial(eA\partial B) \quad (\text{VI.2.29})$$

$$= \pm\partial(eB)\partial(A) \pm \partial(eA)\partial B \quad (\text{since } \partial^2 = 0) \quad (\text{VI.2.30})$$

$$\in (\partial(eA), \partial(eB)) \quad (\text{VI.2.31})$$

□

*Proof of Proposition VI.2.3.* By Lemma VI.2.5, for the arrangement  $\mathcal{A} := \{\Delta_{ij}, \Delta_i^s : 1 \leq i \neq j \leq n, 1 \leq s \leq r\}$ , the algebra  $E_1(\mathcal{A})_{\mathbb{Q}}$  is given by

$$\begin{aligned} E_1(X_r, n) &= \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F)} \\ &= \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F g_S : \bigcap S = F \text{ and } S \text{ is independent})} \end{aligned}$$

We denote the generators of  $A(\mathcal{A})$  by  $g_{ij} = g_{\Delta_{ij}}$ ,  $g_i^s = g_{\Delta_i^s}$  (so that  $g_{ij} = g_{ji}$  are the same generator). We work out the relation ideal  $(I_F g_S : \bigcap S = F \text{ and } S \text{ is independent})$  first. We claim that it is enough to use  $F$  of rank one. In other words, the relation ideal is equal to the ideal  $J$  generated by

(a)  $g_{ij}(\alpha_i - \alpha_j)$ ;

(b)  $g_i^s \alpha_i, \alpha \in H^{\geq 1}(X)$ .

Let  $F = F_{\chi, \sim}$  be a stratum of  $\mathcal{A}$ . We need to show that for any independent  $S \subset \mathcal{A}$  such that  $\bigcap S = F$ , the ideal  $g_S I_F$  is in the ideal  $J$ .

Such an  $S$  is classified by the following indirected graph, consisting of

(a) An (unrooted) spanning tree on each equivalence class of  $\sim$  on  $\chi^{-1}(0)$ ;

(b) A forest of rooted trees on  $\chi^{-1}(s)$ , for each  $1 \leq s \leq r$ .

The set  $S$  then consists of  $\Delta_i^{\chi(i)}$  for each  $i$  that appears as a root and  $\Delta_{ij}$  for each  $(i, j)$  that appears as an edge.

We observe that to generate  $I_F = (\alpha_i - \alpha_j \ (i \sim j), \alpha_i \ (\chi(i) \neq 0))$ , a part of the generators suffices:  $\alpha_i - \alpha_j$  for  $\Delta_{ij} \in S$  and  $\alpha_i$  for  $\Delta_i^s \in S, \alpha \in H^{\geq 1}(X)$ . Indeed, this can be done by joining a path from  $i$  to  $j$  in the tree (if  $i \sim j \in \chi^{-1}(0)$ ) or by joining a path from  $i$  to the root of the tree where  $i$  belongs (if  $\chi(i) \neq 0$ ). But  $g_S$  multiplied by each of these special generators lies in  $J$ . This proves the claim and finishes the computation of  $(I_F \cdot A_F(\mathcal{A}) : F \in L(\mathcal{A}))$ .

It remains to compute a presentation of  $A(\mathcal{A})$ . Let  $J(\mathcal{A})$  be the ideal of  $B(\mathcal{A})$  generated by relations (VI.2.9), (VI.2.12) and (VI.2.13) of Proposition VI.2.3. Claim that  $J(\mathcal{A}) = I(\mathcal{A})$ , the defining ideal for  $A(\mathcal{A})$ .

The ideal  $I(\mathcal{A})$  is generated by  $e_S$  for minimal vanishing set  $e_S$  and  $\partial e_S$  for minimal dependent set  $S$ . These include

- (1)  $e_i^s e_j^t e_\gamma, s \neq t$ ;
- (2)  $\partial(e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{h-1} i_h} e_{i_h i_1})$ , with  $h \geq 3$  and  $i_1, \dots, i_h$  are distinct;
- (3)  $\partial(e_i^s e_j^s e_\gamma)$ ,

where in both (1) and (3),  $\gamma = (i = i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_h = j)$  is a path joining  $i$  and  $j$ , and  $e_\gamma$  means  $e_{i_0 i_1} e_{i_1 i_2} \cdots e_{i_{h-1} i_h}$ . Here  $h$  is allowed to be 0, in which case  $i = j$  and  $e_\gamma = 1$ .

We need to show that these generators are in  $J(\mathcal{A})$ .

First, we prove that (2) is in  $J(\mathcal{A})$  by induction on  $h$ . If  $h = 3$ , then (2) is just (VI.2.9). If  $h > 3$ , we set  $S_1 = \{\Delta_{i_1 i_2}, \Delta_{i_2 i_3}, \dots, \Delta_{i_{h-2} i_{h-1}}\}, S_2 = \{\Delta_{i_{h-1} i_h}, \Delta_{i_h i_1}\}$  and  $Y = \Delta_{i_1 i_{h-1}}$ , then  $\partial e_{S_1 \sqcup Y} \in J(\mathcal{A})$  by induction hypothesis,  $\partial e_{S_2 \sqcup Y} \in J(\mathcal{A})$  by  $h = 3$  case. Applying Lemma VI.2.6, we get  $\partial e_{S_1 \sqcup S_2} \in J(\mathcal{A})$ .

Next, we prove that (3) is in  $J(\mathcal{A})$  by induction on  $h$ , the length of  $\gamma = (i = i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_h = j)$ . The base case  $h = 1$  is (VI.2.12), and the induction step is proved similarly with  $S_1 = \{\Delta_i^2, \Delta_{i_0 i_1}, \Delta_{i_1 i_2}, \dots, \Delta_{i_{h-2} i_{h-1}}\}, S_2 = \{\Delta_{i_{h-1} i_h}, \Delta_j^s\}$  and  $Y = \Delta_{i_{h-1}}^s$ .

Finally, for (1), to prove that  $e_i^s e_j^t e_\gamma \in J(\mathcal{A})$  for  $s \neq t$ , we note that

$$\partial(e_i^t e_j^t e_\gamma) = e_j^t e_\gamma - e_i^t \partial(e_i^t e_\gamma) \quad (\text{VI.2.32})$$

is just proved to be in  $J(\mathcal{A})$  by case (3). Hence

$$e_j^t e_\gamma \equiv e_i^t \partial(e_i^t e_\gamma) \pmod{J(\mathcal{A})}, \quad (\text{VI.2.33})$$

so

$$e_i^s e_j^t e_\gamma \equiv e_i^s e_i^t \partial(e_i^t e_\gamma) \pmod{J(\mathcal{A})}. \quad (\text{VI.2.34})$$

But  $e_i^s e_i^t$  is just (VI.2.13), so  $e_i^s e_j^t e_\gamma \in J(\mathcal{A})$ .

In summary, we have proved that

$$E_1(X_r, n) = \frac{H^*(X^n) \otimes A(\mathcal{A})}{(I_F \cdot A_F(\mathcal{A}) : F)} \quad (\text{VI.2.35})$$

$$= \frac{H^*(X^n) \otimes \mathbb{Q}[g_{ij}, g_i^s] / ((\text{VI.2.8}), (\text{VI.2.9}), (\text{VI.2.12}), (\text{VI.2.13}))}{((\text{VI.2.10}), (\text{VI.2.11}))} \quad (\text{VI.2.36})$$

$$= \frac{H^*(X^n)[g_{ij}, g_i^s]}{((\text{VI.2.8}) \text{ through } (\text{VI.2.13}))} \quad (\text{VI.2.37})$$

This proves the description of  $E_1(X_r, n)$  in Proposition VI.2.3.  $\square$

### VI.3: Proof of Theorem VI.1.5

Let  $\bar{X}$  be a connected smooth complex variety, and let  $P^1, \dots, P^r$  be distinct points of  $\bar{X}$ , where  $r \geq 1$ . Let  $X = \bar{X} - \{P^1, \dots, P^{r-1}\}$  and  $P = P^r$ . Noting that  $X = \bar{X}_{r-1}$  and  $X - P = \bar{X}_r$  in the notation of Proposition VI.2.3, we construct spectral sequences

$$E_1(X, n) := E_1(\bar{X}_{r-1}, n) \implies H^{i-j}(F(X, n); \mathbb{Q}) \quad (\text{VI.3.1})$$

and

$$E_1(X - P, n) := E_1(\bar{X}_r, n) \implies H^{i-j}(F(X - P, n); \mathbb{Q}) \quad (\text{VI.3.2})$$

based on Proposition VI.2.3. We shall prove that these two spectral sequences satisfy the statements of Theorem VI.1.5.

#### VI.3.1: Proof of Theorem VI.1.5(a)

Using the description of  $E_1(\bar{X}_r, n)$  and  $E_1(\bar{X}_{r-1}, n)$  in Proposition VI.2.3 as differential graded algebras, we can express  $E_1(X - P, n)$  as an algebra over  $E_1(X, n)$  according to

$$E_1(X - P, n) = E_1(X, n)[g_1^r, \dots, g_n^r] / (\text{new relations}) \quad (\text{VI.3.3})$$

where the new relations consist of

$$g_i^r g_j^r - g_{ij} g_i^r + g_{ij} g_j^r = 0 \text{ for } 1 \leq i, j \leq n \quad (\text{VI.3.4})$$

$$g_i^s g_i^r = 0 \text{ for } 1 \leq s \leq r - 1 \quad (\text{VI.3.5})$$

$$g_i^r \alpha_i = 0 \text{ for } \alpha \in H^p(\bar{X}), p \geq 1 \quad (\text{VI.3.6})$$

We will repetitively use the following elementary fact about quadratic algebras. Let  $R$

be a graded-commutative ring with identity, and let  $x_1, \dots, x_m$  be indeterminates of degree one. Consider the graded-commutative  $R$ -algebra  $A$  generated by  $x_1, \dots, x_m$  with relations  $x_i x_j = L_{ij}(x_1, \dots, x_m)$  for all  $1 \leq i < j \leq m$ , where  $L_{ij}(x_1, \dots, x_m)$  is a left  $R$ -linear combination of  $x_1, \dots, x_m$ . Then  $A$  is isomorphic to  $R\langle 1, x_1, \dots, x_m \rangle$  as a left  $R$ -module, where  $R\langle 1, x_1, \dots, x_m \rangle$  denotes the free left  $R$ -module with basis  $1, x_1, \dots, x_m$ .

We denote by  $[n]$  the finite set  $\{1, \dots, n\}$ , and by  $[n] - i$  the set  $\{j \in [n] : j \neq i\}$ . For any finite set  $I$  of integers and for  $Y = X$  or  $X - P$ , we denote by  $E_1(Y, I)$  a copy of  $E_1(Y, |I|)$ , but with lower indices of the generators taken from  $I$  instead of  $\{1, \dots, |I|\}$ . Note that  $E_1(Y, n)$  and  $E_1(Y, [n])$  are precisely the same.

We now express  $E_1(X - P, n)$  as a left module over  $E_1(X, n)$ . Note that the relation (VI.3.4) is equivalent to  $g_i^r g_j^r = g_{ij} g_i^r - g_{ij} g_j^r$ , the right-hand side being a linear combination of  $g_i^r, g_j^r$  with coefficients  $g_{ij}$  in the ring  $E_1(X, n)$ . Thus  $E_1(X, n)[g_i^r : i \in [n]]/(VI.3.4)$  is a quadratic algebra over  $E_1(X, n)$ , and we have

$$\frac{E_1(X, n)[g_i^r : i \in [n]]}{(VI.3.4)} \cong E_1(X, n)\langle 1, g_i^r : i \in [n] \rangle \quad (VI.3.7)$$

as a left  $E_1(X, n)$ -module, while its ring structure is given by the multiplication table

$$g_i^r g_j^r = g_{ij} g_i^r - g_{ij} g_j^r. \quad (VI.3.8)$$

Therefore,

$$E_1(X - P, n) = \frac{E_1(X, n)[g_i^r : i \in [n]]/(VI.3.4)}{((VI.3.5), (VI.3.6))} \quad (VI.3.9)$$

$$= \frac{E_1(X, n)\langle 1, g_i^r : i \in [n] \rangle}{(g_i^s g_i^r, \alpha_i g_i^r : i \in [n], s \neq r, \alpha \in H^{\geq 1}(X))} \quad (VI.3.10)$$

$$= E_1(X, n) \oplus \bigoplus_{i=1}^n E_1(X, n)\langle g_i^r \rangle / (g_i^s g_i^r, \alpha_i g_i^r : s \neq r) \quad (VI.3.11)$$

Our next goal is to further decompose each summand of (VI.3.11). Fix  $i \in [n]$ , and we shall compute the presentation  $E_1(X, n)$  as an algebra over  $E_1(X, [n] - i)$ . Comparing the presentations of  $E_1(X, n)$  and  $E_1(X, [n] - i)$  in Proposition VI.2.3, we get the following presentation, where the subscripts  $j, k$  always range over  $[n] - i$ , the superscripts  $s, t$  always range over  $\{1, \dots, r - 1\}$ , and  $\alpha$  is taken from  $H^{\geq 1}(\overline{X})$ .

$$E_1(X, n) = \frac{E_1(X, [n] - i)[g_{ij}, g_i^s] \otimes_{\mathbb{Q}} H^*(X)}{\left( \begin{array}{l} g_{ij}g_{ik} = -g_{jk}g_{ij} + g_{jk}g_{ik}, \quad j \neq k \\ g_{ij}g_i^s = -g_j^s g_i^s - g_j^s g_{ij}, \quad j, s \\ g_i^s g_i^t = 0, \quad s \neq t \\ \alpha_i g_{ij} = \alpha_j g_{ij}, \quad j, \alpha \\ g_i^s \alpha_i = 0, \quad s, \alpha \end{array} \right)} \quad (\text{VI.3.12})$$

where  $H^*(X)$  in the tensor factor contributes to the  $i$ -th coordinate, namely,  $\{\alpha_i : \alpha \in H^*(\overline{X})\}$ . From now on, every tensor product is over  $\mathbb{Q}$ .

We note that if we only take the first three relations in (VI.3.12), we get a quadratic algebra over  $E_1(X, [n] - i) \otimes H^*(X)$ . Thus

$$E_1(X, n) = \frac{(E_1(X, [n] - i) \otimes H^*(\overline{X})) \langle 1, g_{ij}, g_i^s \rangle}{((\alpha_i - \alpha_j)g_{ij}, \alpha_i g_i^s)}. \quad (\text{VI.3.13})$$

Recall that for any  $\alpha$  in  $H^{\geq 1}(\overline{X})$ , the element  $\alpha_i$  of  $E_1(X, n)$  is understood as the tensor  $1 \otimes \alpha$  in  $E_1(X, [n] - i) \otimes H^*(\overline{X})$ , and the element  $\alpha_j$  is understood as the tensor  $\alpha_j \otimes 1$  in  $E_1(X, [n] - i) \otimes H^*(\overline{X})$ . Recalling that  $H^*(\overline{X}) = \mathbb{Q} \oplus H^{\geq 1}(\overline{X})$ , we may decompose the numerator of (VI.3.13) as

$$\begin{aligned} (E_1(X, [n] - i) \otimes H^*(\overline{X})) \langle 1, g_{ij}, g_i^s \rangle &= E_1(X, [n] - i) \otimes \\ &\left( H^*(\overline{X}) \oplus \bigoplus_j \mathbb{Q}g_{ij} \oplus \bigoplus_j H^{\geq 1}(\overline{X})g_{ij} \oplus \bigoplus_s \mathbb{Q}g_i^s \oplus \bigoplus_s H^{\geq 1}(\overline{X})g_i^s \right). \end{aligned} \quad (\text{VI.3.14})$$

The relation  $(\alpha_i - \alpha_j)g_{ij} = 0$  in (VI.3.13) identifies a general element  $\alpha_i g_{ij}$  of the summand  $H^{\geq 1}(\overline{X})g_{ij}$  with the element  $\alpha_j g_{ij}$  of  $E_1(X, [n] - i) \otimes \mathbb{Q}g_{ij}$ . As a result, the relation  $(\alpha_i - \alpha_j)g_{ij}$  kills the summand  $H^{\geq 1}(\overline{X})g_{ij}$  without introducing other identifications. The same argument can be applied to the relation  $\alpha_i g_i^s = 0$ . It follows that

$$E_1(X, n) = \frac{(E_1(X, [n] - i) \otimes H^*(\overline{X})) \langle 1, g_{ij}, g_i^s \rangle}{((\alpha_i - \alpha_j)g_{ij}, \alpha_i g_i^s)} \quad (\text{VI.3.15})$$

$$= E_1(X, [n] - i) \otimes (H^*(\overline{X}) \oplus \mathbb{Q} \langle g_{ij}, g_i^s \rangle) \quad (\text{VI.3.16})$$

$$= E_1(X, [n] - i) \otimes (H^{\geq 1}(\overline{X}) \oplus \mathbb{Q} \langle 1, g_{ij}, g_i^s \rangle). \quad (\text{VI.3.17})$$

as a module over  $E_1(X, [n] - i)$ , and the ring structure of  $E_1(X, n)$  can be read from this representation with a multiplication table given by the relations in (VI.3.12).

We are now ready to give a presentation of each summand of (VI.3.11). Fix  $i$ . In the

computation below, the convention for  $\alpha, j, s, t$  is as before, and  $\beta$  ranges over  $H^{\geq 1}(X)$ .

$$\frac{E_1(X, n)\langle g_i^r \rangle}{E_1(X, n)(g_i^s g_i^r, \alpha_i g_i^r : s, \alpha)} = \frac{E_1(X, [n] - i) \otimes (H^{\geq 1}(\overline{X}) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s \rangle)}{E_1(X, [n] - i)(\beta_i, 1, g_{ij}, g_i^t)(g_i^s, \alpha_i)} \langle g_i^r \rangle \quad (\text{VI.3.18})$$

$$= \frac{E_1(X, [n] - i) \otimes (H^{\geq 1}(\overline{X}) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s \rangle)}{E_1(X, [n] - i) \begin{pmatrix} g_i^s \beta_i & g_i^s & g_i^s g_{ij} & g_i^s g_i^t \\ \alpha_i \beta_i & \alpha_i & \alpha_i g_{ij} & \alpha_i g_i^t \end{pmatrix}} \langle g_i^r \rangle \quad (\text{VI.3.19})$$

$$\stackrel{(\text{VI.3.12})}{=} \frac{E_1(X, [n] - i) \otimes (H^{\geq 1}(\overline{X}) \oplus \mathbb{Q}\langle 1, g_{ij}, g_i^s \rangle)}{E_1(X, [n] - i) \begin{pmatrix} 0 & g_i^s & g_j^s g_i^s + g_j^s g_{ij} & 0 \\ (\alpha\beta)_i & \alpha_i & \alpha_j g_{ij} & 0 \end{pmatrix}} \langle g_i^r \rangle \quad (\text{VI.3.20})$$

$$= \frac{E_1(X, [n] - i)_{\mathbb{Q}} \langle 1, g_{ij} : j \in [n] - i \rangle}{(g_j^s g_{ij}, \alpha_j g_{ij} : j \in [n] - i, s \neq r, \alpha \in H^{\geq 1}(\overline{X}))} \langle g_i^r \rangle \quad (\text{VI.3.21})$$

$$= \frac{E_1(X, [n] - i) \langle g_r, g_{ij} g_r : j \in [n] - i \rangle}{(g_j^s g_{ij} g_r, \alpha_j g_{ij} g_r : j \in [n] - i, s \neq r, \alpha \in H^{\geq 1}(\overline{X}))} \quad (\text{VI.3.22})$$

where we note that the generators  $g_i^s$  and the summand  $H^{\geq 1}(X)$  are eliminated by the relations. In line (VI.3.22), we view  $g_r$  and  $g_{ij} g_r$  as two separate formal generators.

We now prove Theorem VI.1.5(a). The important observation is that the presentation (VI.3.22) is isomorphic to the presentation of  $E_1(X - P, [n] - i)$  given by (VI.3.10) applied to the index set  $[n] - i$ :

$$E_1(X - P, [n] - i) = \frac{E_1(X, [n] - i) \langle 1, g_j^r : j \in [n] - i \rangle}{(g_j^s g_j^r, \alpha_j g_j^r : j \in [n] - i, s \neq r, \alpha \in H^{\geq 1}(\overline{X}))}, \quad (\text{VI.3.23})$$

where the correspondence is given by  $g_r \mapsto 1$  and  $g_{ij} g_r \mapsto g_j^r$ .

Putting the summands of (VI.3.11) together, we finish construction of the required isomorphism  $\Phi$ , as is stated below.

**Lemma VI.3.1.** *There exists a  $\mathbb{Q}$ -linear isomorphism*

$$\Phi : E_1(X, n) \oplus \bigoplus_{i=1}^n E_1(X - P, [n] - i) \rightarrow E_1(X - P, n), \quad (\text{VI.3.24})$$

such that  $\Phi|_{E_1(X, n)} : E_1(X, n) \rightarrow E_1(X - P, n)$  is the natural ring homomorphism, and  $\Phi|_{E_1(X - P, [n] - i)}$  is the  $E_1(X, [n] - i)$ -linear map that sends 1 to  $g_i^r$  and sends  $g_j^r$  to  $g_{ij} g_i^r$  for all  $j \in [n] - i$ .

We note that  $\bigoplus_{i=1}^n E_1(X - P, [n] - i)$  is isomorphic to  $\text{Ind}_{S_{n-1}}^{S_n} E_1(X - P, n - 1)$  as an

$S_n$ -module. Indeed, the induction of  $E_1(X - P, n - 1)$  from  $S_{n-1}$  to  $S_n$  is the direct sum of images of  $E_1(X - P, n - 1)$  under cosets of  $S_{n-1}$  in  $S_n$ . There are  $n$  cosets, and for each  $i \in [n]$ , there is precisely one coset such that every element in it maps  $E_1(X - P, n - 1)$  onto  $E_1(X - P, [n] - i)$ . Hence the induction of  $E_1(X - P, n - 1)$  is the direct sum of  $E_1(X - P, [n] - i)$  for  $i \in [n]$ .

Note that the elements  $g_i^r$  and  $g_{ij}$  have bidegree  $(2d, 1)$  and Hodge type  $(d, d)$  in the dga  $E_1(X - P, n)$ . Keeping track of the bidegree shifting and the Tate twist, and noting that  $S_n$  acts by permuting the subscripts, we obtain an isomorphism

$$\Phi : E_1^{i,j}(X, n) \oplus \text{Ind}_{S_{n-1}}^{S_n} E_1^{i-2d, j-1}(X - P, n - 1)(-d) \rightarrow E_1^{i,j}(X - P, n) \quad (\text{VI.3.25})$$

as  $S_n$ -mixed Hodge structures, which finishes the proof of Theorem VI.1.5(a).

### VI.3.2: Proof of Theorem VI.1.5(b)

Now assume that  $\bar{X}$  is a  $d$ -dimensional smooth complex variety that is not compact. We need to show that  $\Phi$  constructed above commutes with the differential.

Recall that the restriction of  $\Phi$  on the  $i$ -th summand is given by an  $E_1(X, [n] - i)$ -linear map

$$\Phi_i : E_1(X - P, [n] - i) \rightarrow E_1(X - P, n) \quad (\text{VI.3.26})$$

such that  $\Phi_i(1) = g_i^r$  and  $\Phi_i(g_j^r) = g_{ij}g_i^r$  for  $j \in [n] - i$ . It suffices to show that  $d(\Phi_i(1)) = \Phi_i(d(1))$  and  $d(\Phi_i(g_j^r)) = \Phi_i(d(g_j^r))$ .

We claim that as elements of  $E_1(X - P, n)$ , we have  $dg_i^r = 0$  and  $d(g_{ij}g_i^r) = 0$  for  $j \in [n] - i$ .

Since  $\bar{X}$  is not compact, the top cohomology  $H^{2d}(\text{bbar}X)$  vanishes. We have  $dg_i^r = [\Delta_i^r] = p_i^*([P^r])$  in the notation of Section VI.2.3. But  $[P^r]$  lies in  $H^{2d}(\bar{X})$  and  $H^{2d}(\bar{X}) = 0$ , so  $dg_i^r = 0$  for all  $i \in [n]$ .

To compute  $d(g_{ij}g_i^r)$ , we use the graded Leibniz rule, noting that  $dg_i^r = 0$  just obtained above.

$$d(g_{ij}g_i^r) = d(g_{ij})g_i^r - g_{ij}dg_i^r = d(g_{ij})g_i^r = p_{ij}^*[\Delta]g_i^r. \quad (\text{VI.3.27})$$

Note that  $[\Delta] \in H^{2d}(\bar{X} \times \bar{X})$ , but  $H^{2d}(\bar{X}) = 0$ , so Künneth's formula implies that  $[\Delta] \in \bigoplus_{p=1}^{2d-1} H^p(\bar{X}) \otimes H^{2d-p}(\bar{X})$ . In particular,  $p_{ij}^*[\Delta]$  can be expressed as a  $\mathbb{Q}$ -linear combination of terms of the form  $\alpha_i\beta_j$ , where  $\alpha, \beta \in H^{\geq 1}(\bar{X})$ . Since  $\alpha_i g_i^r = 0$  in  $E_1(X - P, n)$  for every  $\alpha \in H^{\geq 1}(\bar{X})$  (Relation (VI.2.11)), we see that  $d(g_{ij}g_i^r) = 0$  in  $E_1(X - P, n)$ .

We now verify that  $\Phi$  commutes with the differential. Since  $dg_i^r = 0$ , the left-hand side of  $d(\Phi_i(1)) = \Phi_i(d(1))$  is zero, verifying the equality. Note that we have proved  $dg_i^r = 0$  in



$E_1(X - P, [n])$  for all  $i \in [n]$ . Applying this fact to the index set  $[n] - i$ , we have  $dg_j^r = 0$  for  $j \in [n] - i$ . Hence, the right-hand side of  $d(\Phi_i(g_j^r)) = \Phi_i(d(g_j^r))$  is zero. But the left-hand side is  $d(g_{ij}g_i^r)$ , which we have computed to be zero as well. This finishes the verification of the equalities, and hence the proof of Theorem VI.1.5(b).

### VI.3.3: Proof of Theorem VI.1.5(c)

Recall that the spectral sequences  $E(X - P, n)$  and  $E(X, n)$  are constructed as  $E(\overline{X}_r, n)$  and  $E(\overline{X}_{r-1}, n)$ , respectively, in Proposition VI.2.3. Therefore, it suffices to show that for any connected smooth complex variety  $X$  that is pure of slope  $\lambda$  and any integer  $r \geq 0$ , the spectral sequence  $E(X_r, n)$  degenerates on  $E_2$ . (For convenience purposes, we use  $X$  in place of  $\overline{X}$ .)

For any  $k, l \geq 0$ , the bidegree- $(k, l)$  part  $E_1^{k,l}(X_r, n)$  of  $E_1(X_r, n)$  is generated by the product of  $l$  Orlik–Solomon generators (denoted  $g_{ij}$  and  $g_i^s$  in Proposition VI.2.3, each with bidegree  $(2d, 1)$  and Hodge type  $(d, d)$ ) and an element of  $H^{k-2dl}(X^n)$ . By Künneth’s formula,  $X^n$  is pure of slope  $\lambda$  as well, so  $H^{k-2dl}(X^n)$  is pure of weight  $\lambda(k - 2dl)$ . Each of the  $l$  Orlik–Solomon generators has a weight of  $d + d = 2d$ , so that

$$E_1^{k,l}(X_r, n) \text{ is pure of weight } \lambda(k - 2dl) + l(2d). \quad (\text{VI.3.28})$$

The same purity holds for any higher page  $E_h^{k,l}(X_r, n)$  (where  $h \geq 2$ ) as well, since  $E_h$  is a subquotient of  $E_1$  from the general definition of spectral sequences.

We now show that the spectral sequence  $E(X_r, n)$  degenerates on  $E_2$  for weight reason. Suppose the differential map

$$d_h : E_h^{k,l}(X_r, n) \rightarrow E_h^{k+h-1, l-h}(X_r, n) \quad (\text{VI.3.29})$$

is nonzero for some  $h \geq 2$  and  $k, l \geq 0$ . Then  $E_h^{k,l}(X_r, n)$  and  $E_h^{k+h-1, l-h}(X_r, n)$  must be nonzero pure Hodge structures with equal weight, since the differentials are strictly compatible with the weight filtration. From (VI.3.28), we get

$$\lambda(k - 2dl) + 2dl = \lambda((k + h - 1) - 2d(l - h)) + 2d(l - h), \quad (\text{VI.3.30})$$

which simplifies to

$$\lambda = \frac{2dh}{2dh + h - 1}. \quad (\text{VI.3.31})$$

Since  $h > 1$ , we have

$$\frac{2dh}{2dh + h - 1} < 1, \quad (\text{VI.3.32})$$

a contradiction to the fact that  $1 \leq \lambda \leq 2$ . This finishes the proof of Theorem VI.1.5(c).

*Remark VI.3.2.* This weight argument is well-known in the case  $\lambda = 1$ ; see for example [Tot96] and [Bib16].

*Remark VI.3.3.* In fact, if  $\lambda > 1$ , then (VI.3.31) is a contradiction even when  $h = 1$ , so the spectral sequence  $E(X_r, n)$  degenerates already on the  $E_1$  page.

## VI.4: Proof of Theorem VI.1.4

Let  $\bar{X}$  and  $X$  be as in the assumption of Theorem VI.1.3. By Theorem VI.1.5, we have spectral sequences of  $S_n$ -mixed Hodge structures  $E_1^{i,j}(X - P, n) \implies H^{i-j}(F(X - P, n))$  and  $E_1^{i,j}(X, n) \implies H^{i-j}(F(X, n))$  that degenerate on the  $E_2$  page. We have also an isomorphism of  $S_n$ -mixed Hodge structures

$$\Phi : E_1^{i,j}(X, n) \oplus \text{Ind}_{S_{n-1}}^{S_n} E_1^{i-2d, j-1}(X - P, n - 1)(-d) \rightarrow E_1^{i,j}(X - P, n) \quad (\text{VI.4.1})$$

that commutes with the differential map. Because of its commutativity with the differential, the isomorphism  $\Phi$  descends to an isomorphism on the  $E_2$  page, which is the same as the  $E_\infty$  page due to degeneracy. We get an isomorphism of  $S_n$ -mixed Hodge structures

$$\Phi_\infty^{i,j} : E_\infty^{i,j}(X, n) \oplus \text{Ind}_{S_{n-1}}^{S_n} E_\infty^{i-2d, j-1}(X - P, n - 1)(-d) \rightarrow E_\infty^{i,j}(X - P, n). \quad (\text{VI.4.2})$$

Fix  $k \in \mathbb{Z}$ . The convergence of the spectral sequence  $E_1^{i,j}(X, n) \implies H^{i-j}(F(X, n))$  means that there is a filtration of  $H^k(F(X, n))$  whose successive quotients consist of  $E_\infty^{k+t, t}(X, n)$  for all  $t \in \mathbb{Z}$ . In particular, if we apply the exact functor  $\text{gr}$  from the category of  $(S_n)$ -mixed Hodge structures to the category of  $(S_n)$ -split mixed Hodge structures, defined by taking the associated graded according to the weight filtration, we get a noncanonical isomorphism

$$\text{gr } H^k(F(X, n)) \cong \bigoplus_{t \in \mathbb{Z}} \text{gr } E_\infty^{k+t, t}(X, n), \quad (\text{VI.4.3})$$

since the category of  $(S_n)$ -split mixed Hodge structures is semisimple. Similarly for  $F(X - P, n)$ .

Summing up the isomorphisms  $\Phi_\infty^{k+t, t}$  for all  $t \in \mathbb{Z}$  and applying the  $\text{gr}$  functor, we get

(working from the right-hand side)

$$\mathrm{gr} H^k(F(X - P, n)) \cong \bigoplus_{t \in \mathbb{Z}} \mathrm{gr} E_\infty^{k+t, t}(X - P, n) \quad (\text{VI.4.4})$$

$$\cong \bigoplus_{t \in \mathbb{Z}} \left( \mathrm{gr} E_\infty^{k+t, t}(X, n) \oplus \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{gr} E_\infty^{k+t-2d, k+t-1}(X - P, n - 1)(-d) \right) \quad (\text{VI.4.5})$$

$$\cong \mathrm{gr} H^k(F(X, n)) \oplus \mathrm{Ind}_{S_{n-1}}^{S_n} \mathrm{gr} H^{k-2d, k-1}(F(X - P, n - 1))(-d). \quad (\text{VI.4.6})$$

By repetitively applying the above isomorphism, or by induction on  $n$ , we get

$$\mathrm{gr} H^k(F(X - P, n), \mathbb{Q}) \cong \bigoplus_{t=0}^{\infty} \mathrm{Ind}_{S_{n-t}}^{S_n} \mathrm{gr} H^{k-(2d-1)t}(F(X, n - t), \mathbb{Q})(-d \cdot t), \quad (\text{VI.4.7})$$

finishing the proof of (VI.1.15). By forgetting the mixed Hodge structures and noting that the category of finite-dimensional  $S_n$ -representations over  $\mathbb{Q}$  is semisimple, we complete the proof of (VI.1.14) and thus Theorem VI.1.4. As is explained in Section VI.1.4, Theorem VI.1.4 implies Theorem VI.1.1 and Theorem VI.1.3, thus the proof of all results is complete.

## VI.5: Examples and discussions

We discuss several examples of connected smooth noncompact complex varieties that are pure of a certain slope, and therefore satisfy the assumptions for  $\overline{X}$  in Theorem VI.1.3 and VI.1.4. Recall that a complex variety  $\overline{X}$  is pure of slope  $\lambda$  if the mixed Hodge structure on  $H^i(\overline{X}; \mathbb{Q})$  is pure of weight  $\lambda \cdot i$  whenever  $\lambda \cdot i$  is an integer, and  $H^i(\overline{X}; \mathbb{Q})$  is zero whenever  $\lambda \cdot i$  is not an integer.

- (a) Let  $\overline{X}$  be the complement of a hyperplane arrangement in  $\mathbb{C}^d$  or a toric arrangement in  $(\mathbb{C}^*)^d$ . Then  $H^i(\overline{X}; \mathbb{Q})$  is pure of weight  $2i$  by [Dup16, Theorems 3.7, 3.8]. In other words,  $\overline{X}$  is pure of slope 2.
- (b) Let  $C$  be a smooth projective curve of genus  $g$  embedded in the projective plane  $\mathbb{P}^2$ , and let  $\overline{X} = \mathbb{P}^2 - C$ . We have

$$H^i(\overline{X}; \mathbb{Q}) = \begin{cases} \mathbb{Q}, \text{ pure of weight } 0, & i = 0 \\ \mathbb{Q}^{2g}, \text{ pure of weight } 3, & i = 2 \\ 0, & i \neq 0, 2. \end{cases} \quad (\text{VI.5.1})$$

Hence  $\overline{X}$  is pure of weight  $3/2$ .

- (c) Suppose  $\overline{X}$  is smooth and pure of slope  $\lambda$ , and  $G$  is a finite group that acts on  $\overline{X}$  freely such that the scheme-theoretic quotient  $X/G$  is also smooth. Then  $H^i(\overline{X}; \mathbb{Q})$  is pure of weight  $\lambda \cdot i$  for all  $i$ . Since  $H^i(\overline{X}/G; \mathbb{Q})$  is the  $G$ -invariant of  $H^i(\overline{X}; \mathbb{Q})$ , we have that  $H^i(\overline{X}/G; \mathbb{Q})$  is also pure of weight  $\lambda \cdot i$ . Thus  $\overline{X}/G$  is also pure of slope  $\lambda$ . As an application, the generalized configuration space  $F(\mathbb{C}, m)/G$  for a subgroup  $G$  of  $S_m$  is pure of slope  $2$ .

We formulate the following questions in an attempt to generalize the formula (VI.1.11), and consequently Kallel's (VI.1.7), beyond the already flexible family of examples of  $X$  described in Theorem VI.1.3. We emphasize that both formulas are known to be false if  $X$  is compact.

**Question VI.5.1.**

- (a) Does the conclusion of Theorem VI.1.1 hold for any connected noncompact smooth complex variety?
- (b) If yes, how about the refined statements in Theorem VI.1.4?

The difficulty of further generalizing the main theorems lies in the Hodge-theoretic hypothesis of Theorem VI.1.5(c). The isomorphism  $\Phi$  in Theorem VI.1.5(a) is constructed “artificially” based on matching explicit generators and relations. It is unlikely that  $\Phi$  has an equivalent construction that is functorial, because  $\Phi$  does not necessarily commute with the differential maps if  $\overline{X}$  is compact, as is clear from the proof of Theorem VI.1.5(b). Therefore, we do not have control of  $\Phi$  on higher pages of the spectral sequences, so the degeneracy statement in Theorem VI.1.5(c) is necessary to obtain an isomorphism on the  $E_\infty$  page.

We conclude the paper by giving an example of  $\overline{X}$  where the hypothesis of Theorem VI.1.5(c) is false, but it is unknown whether the conclusion of Theorem VI.1.5(c) or Theorem VI.1.3 is true.

**Example VI.5.2.** Let  $E$  be an elliptic curve, and  $O$  be a point of  $E$ . Consider the smooth surface  $\overline{X} = \text{Conf}^2(E - O)$ . As is computed in [CH22], we have

$$H^i(\overline{X}; \mathbb{Q}) = \begin{cases} \mathbb{Q}, \text{ pure of weight } 0, & i = 0 \\ \mathbb{Q}^2, \text{ pure of weight } 1, & i = 1 \\ \mathbb{Q}^2, \text{ pure of weight } 3, & i = 2 \\ 0, & i \geq 3 \end{cases} \quad (\text{VI.5.2})$$

We observe that for  $n \geq 11$  and  $r \geq 0$ , the second-page differential map

$$d_2 : E_2^{10,2}(\overline{X}_r, n) \rightarrow E_2^{11,0}(\overline{X}_r, n) \tag{VI.5.3}$$

is not forced to be zero by the weight argument (and is in fact not yet known to be zero), because each side may have a nonzero weight-11 part. The potentially nonzero weight-11 part of  $E_2^{10,2}(\overline{X}_r, n)$  is essentially the contribution of  $H^2(\overline{X}; \mathbb{Q})$  of weight 3.

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